

C4.3 Functional Analytic Methods for PDEs Lecture 4

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- Differentiation rule for convolution.
- Approximation of identity
 - \star in the space of continuous functions,
 - \star in the space of Lipschitz continuous functions,
 - \star in Lebesgue spaces

- Approximation of identity in Lebesgue spaces.
- Density by smooth functions.
- Pre-compactness criteria.

Theorem (Approximation of identity)

Let ρ be a non-negative function in $L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \rho = 1$. For $\varepsilon > 0$, let

$$\varrho_{\varepsilon}(x) = \frac{1}{\varepsilon^n} \varrho\left(\frac{x}{\varepsilon}\right).$$

If $f \in L^p(\mathbb{R}^n)$ for some $1 \le p < \infty$, then

$$\lim_{\varepsilon\to 0} \|f*\varrho_{\varepsilon}-f\|_{L^p(\mathbb{R}^n)}=0.$$

Proof

• Let
$$f_{\varepsilon}(x) := f * \varrho_{\varepsilon}(x)$$
. Then

$$f_{\varepsilon}(x) := f * \varrho_{\varepsilon}(x) = \int_{\mathbb{R}^n} f(y) \varrho_{\varepsilon}(x-y) \, dy = \int_{\mathbb{R}^n} f(x-y) \varrho_{\varepsilon}(y) \, dy.$$

• Recall that, as $\int_{\mathbb{R}^n} arrho_arepsilon = 1$, we have

$$f(x) = \int_{\mathbb{R}^n} f(x) \varrho_{\varepsilon}(y) \, dy.$$

Hence

$$egin{aligned} |f_arepsilon(x)-f(x)|&\leq \int_{\mathbb{R}^n} |f(x-y)-f(x)||arepsilon_arepsilon(y)|dy\ &=\int_{\mathbb{R}^n} |f(x-y)-f(x)||arepsilon_arepsilon(y)|^{rac{1}{p}}|arepsilon_arepsilon(y)|^{rac{1}{p'}}dy. \end{aligned}$$

Proof

•
$$|f_{\varepsilon}(x) - f(x)| \leq \int_{\mathbb{R}^n} |f(x - y) - f(x)| |\varrho_{\varepsilon}(y)|^{\frac{1}{p}} |\varrho_{\varepsilon}(y)|^{\frac{1}{p'}} dy.$$

• Applying Hölder's inequality, the above is less than or equal to

$$\leq \left\{ \int_{\mathbb{R}^n} |f(x-y) - f(x)|^p |\varrho_{\varepsilon}(y)| \, dy \right\}^{\frac{1}{p}} \left\{ \int_{\mathbb{R}^n} |\varrho_{\varepsilon}(y)| \, dy \right\}^{\frac{1}{p'}} \\ = \left\{ \int_{\mathbb{R}^n} |f(x-y) - f(x)|^p |\varrho_{\varepsilon}(y)| \, dy \right\}^{\frac{1}{p}}.$$

• Integrating and using Tonelli's theorem,

$$\begin{split} \|f_{\varepsilon}-f\|_{L^{p}}^{p} &\leq \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} |f(x-y)-f(x)|^{p} |\varrho_{\varepsilon}(y)| \, dy \, dx \\ &= \int_{\mathbb{R}^{n}} |\varrho_{\varepsilon}(y)| \Big\{ \int_{\mathbb{R}^{n}} |f(x-y)-f(x)|^{p} \, dx \Big\} dy. \end{split}$$

Proof

•
$$||f_{\varepsilon}-f||_{L^p}^p \leq \int_{\mathbb{R}^n} |\varrho_{\varepsilon}(y)| \Big\{ \int_{\mathbb{R}^n} |f(x-y)-f(x)|^p dx \Big\} dy.$$

• In other words,

$$\|f_{\varepsilon}-f\|_{L^p}^p\leq \int_{\mathbb{R}^n}|\varrho_{\varepsilon}(y)|\|\tau_{-y}f-f\|_{L^p}^pdy.$$

• If we had that $Supp(\varrho) \subset B_R$, then $Supp(\varrho_{\varepsilon}) \subset B_{\varepsilon R}$, and so

$$\begin{split} \|f_{\varepsilon} - f\|_{L^{p}}^{p} &\leq \sup_{|y| \leq \varepsilon R} \|\tau_{-y} f - f\|_{L^{p}}^{p} \int_{B_{\varepsilon R}} |\varrho_{\varepsilon}(y)| dy \\ &= \sup_{|y| \leq \varepsilon R} \|\tau_{-y} f - f\|_{L^{p}}^{p} \xrightarrow{\varepsilon \to 0} 0, \end{split}$$

in view of the theorem on the continuity of the translation operator in L^p .

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•
$$\|f_{\varepsilon}-f\|_{L^p}^p\leq \int_{\mathbb{R}^n}|\varrho_{\varepsilon}(y)|\|\tau_{-y}f-f\|_{L^p}^pdy.$$

• In the general case where ρ may or may not have compact support, we argue as follows: For every fixed $\hat{R} > 0$,

$$\begin{split} &\int_{|y|\leq \varepsilon \hat{R}} |\varrho_{\varepsilon}(y)| \|\tau_{-y}f - f\|_{L^{p}}^{p} dy \\ &\leq \sup_{|y|\leq \varepsilon \hat{R}} \|\tau_{-y}f - f\|_{L^{p}}^{p} \int_{B_{\varepsilon \hat{R}}} |\varrho_{\varepsilon}(y)| dy \\ &\leq \sup_{|y|\leq \varepsilon \hat{R}} \|\tau_{-y}f - f\|_{L^{p}}^{p} \stackrel{\varepsilon \to 0}{\longrightarrow} 0. \end{split}$$

Approximation of identity in L^p settings

Proof

- $\|f_{\varepsilon} f\|_{L^p}^p \leq \int_{\mathbb{D}^n} |\rho_{\varepsilon}(y)| \|\tau_{-y}f f\|_{L^p}^p dy.$ • $\forall \hat{R}$, $\lim_{\varepsilon \to 0} \int_{|y| < \varepsilon \hat{R}} |\varrho_{\varepsilon}(y)| \|\tau_{-y}f - f\|_{L^{p}}^{p} dy = 0.$
- On the other hand,

$$egin{aligned} &\int_{|y|\geqarepsilon\hat{\mathcal{R}}}|arrho_arepsilon(y)|\| au_{-y}f-f\|_{L^p}^pdy\ &\leq\int_{|y|\geqarepsilon\hat{\mathcal{R}}}|arrho_arepsilon(y)|(\| au_{-y}f\|_{L^p}+\|f\|_{L^p})^pdy\ &=2^p\|f\|_{L^p}^p\int_{|y|\geqarepsilon\hat{\mathcal{R}}}|arrho_arepsilon(y)|dy\!=2^p\|f\|_{L^p}^p\int_{|z|\geq\hat{\mathcal{R}}}|arrho(z)|dz. \end{aligned}$$

• As $\rho \in L^1(\mathbb{R}^n)$, we have by Lebesgue's dominated convergence theorem that

$$\lim_{\hat{R}\to\infty}\int_{|z|\geq\hat{R}}|\varrho(z)|dz=0.$$

Proof

- $\|f_{\varepsilon} f\|_{L^p}^p \leq \int_{\mathbb{R}^n} |\varrho_{\varepsilon}(y)| \|\tau_{-y}f f\|_{L^p}^p dy.$
- $\forall \hat{R}$, $\lim_{\varepsilon \to 0} \int_{|y| \le \varepsilon \hat{R}} |\varrho_{\varepsilon}(y)| ||\tau_{-y}f f||_{L^p}^p dy = 0.$
- $\forall \varepsilon$, $\lim_{\hat{R}\to\infty} \int_{|y|\geq\varepsilon\hat{R}} |\varrho_{\varepsilon}(y)| \|\tau_{-y}f f\|_{L^{p}}^{p} dy = 0.$
- We are ready to wrap up the proof: Fix some $\eta > 0$ and select some large \hat{R} so that

$$\int_{|y|\geq \varepsilon \hat{R}} |\varrho_{\varepsilon}(y)| \|\tau_{-y}f - f\|_{L^p}^p dy \leq \eta/2.$$

• Then we select small ε_0 such that, for all $\varepsilon < \varepsilon_0$,

$$\int_{|y|\leq \varepsilon \hat{R}} |\varrho_{\varepsilon}(y)| \|\tau_{-y}f - f\|_{L^{p}}^{p} dy \leq \eta/2.$$

• For $\varepsilon < \varepsilon_0$,

$$\|f_{\varepsilon}-f\|_{L^p}^p\leq \int_{\mathbb{R}^n}|arrho_{\varepsilon}(y)|\| au_{-y}f-f\|_{L^p}^pdy\leq\eta.$$

• As η is arbitrary, we conclude that $\|f_{\varepsilon} - f\|_{L^{p}(\mathbb{R}^{n})} \to 0$ as $\varepsilon \to 0$.

 $C^{\infty}_{c}(\mathbb{R}^{n}) = L^{p}(\mathbb{R}^{n})$

Theorem

For $1 \leq p < \infty$, the space $C_c^{\infty}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

Proof

- Fix $f \in L^{p}(\mathbb{R}^{n})$. We need to produce $f_{k} \in C_{c}^{\infty}(\mathbb{R}^{n})$ such that $f_{k} \to f$ in L^{p} .
- If f has compact support, say Supp(f) ⊂ B_R, this follows from the previous theorem:
 - * Take a smooth non-negative function *ρ* ∈ *C*[∞]_c(*B*₁) with ∫_{ℝⁿ} *ρ* = 1 and define the mollifiers *ρ*_ε(*x*) = ¹/_{εⁿ}*ρ*(*x*/ε).
 * Then *f_k* := *f* * *ρ*_{1/k} ∈ *C*[∞] and *f_k* → *f* in *L^p*.
 * Recall that *f_k(x)* = ∫_{ℝⁿ} *f*(*y*)*ρ*_{1/k}(*x* − *y*) *dy*, and observe that for |*x*| > *R* + 1/*k*, then, by triangle inequality, |*y*| > *R* or |*x* − *y*| > 1/*k*. So *f*(*y*)*ρ*_{1/k}(*x* − *y*) ≡ 0 for those *x*. So Supp(*f_k*) ⊂ *B*_{R+1/k}, and *f_k* ∈ *C*[∞](ℝⁿ).

 $C^{\infty}_{c}(\mathbb{R}^{n}) = L^{p}(\mathbb{R}^{n})$

- In general, we produce $f_k \in C_c^{\infty}(\mathbb{R}^n)$ with $||f_k f||_{L^p} \le 1/k$ as follows:
 - * Note that, as $R \to \infty$, $f \chi_{B_R} \to f$ in L^p by Lebesgue's dominated convergence theorem.
 - * So we can pick large R_k such that $\|f\chi_{B_{R_k}} f\|_{L^p} \leq \frac{1}{2k}$.
 - Now f χ_{B_{Rk}} has compact support, so by the previous consideration, there exists a function f_k ∈ C_c[∞](ℝⁿ) such that ||f χ_{B_{Rk}} - f_k||_{L^p} ≤ 1/k and so f_k → f in L^p as wanted

$$\star$$
 Then $\|f_k - f\|_{L^p} \leq 1/k$ and so $f_k \to f$ in L^p as wanted.

$$\frac{C^{\infty}(E) \cap L^{p}(E)}{\text{Let } C^{\infty}(E) = \left\{ f|_{E} : f \in C^{\infty}(\mathbb{R}^{n}) \right\}}.$$

Theorem

For $1 \le p < \infty$, the space $C^{\infty}(E) \cap L^{p}(E)$ is dense in $L^{p}(E)$.

Proof

Fix f ∈ L^p(E). We will produce functions f_k ∈ C[∞](ℝⁿ) ∩ L^p(ℝⁿ) such that f_k|_E → f in L^p(E).
Extend f to f̃ : ℝⁿ → ℝ by setting f̃ = 0 in ℝⁿ \ E. Then f̃ ∈ L^p(ℝⁿ). By the previous theorem, there exist f_k ∈ C[∞]_c(ℝⁿ) ⊂ C[∞](ℝⁿ) ∩ L^p(ℝⁿ) such that f_k → f̃ in L^p(ℝⁿ).
Now,

$$\int_E |f_k-f|^p \, dx \leq \int_{\mathbb{R}^n} |f_k-\tilde{f}|^p \, dx \to 0,$$

and so $f_k|_E \to f$ in $L^p(E)$.

Theorem (Ascoli-Arzelà's theorem)

Let K be a compact subset of \mathbb{R}^n . Suppose that (f_i) is a sequence of functions of C(K) such that

- (Boundedness) $\sup_i ||f_i||_{C(K)} < \infty$,
- (Equi-continuity) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f_i(x) f_i(y)| < \varepsilon$ for all i and all $x, y \in K$ with $|x y| < \delta$.

Then there exists a subsequence (f_{i_i}) which converges uniformly on K.

In other words, the set $\{f_i\}$ is pre-compact.

 We would like to show that (f_i) has a subsequence (f_{ij}) which is Cauchy in C(K), i.e. for every given ε > 0,

$$\|f_{i_j} - f_{i_k}\|_{\mathcal{C}(\mathcal{K})} \le \varepsilon \text{ for all large } j, k.$$
 (*)

• We claim that a slightly softer statement holds: For every given ε , there is a subsequence $(f_{i_i}^{\varepsilon})$ of (f_i) such that

$$\|f_{i_j}^{\varepsilon} - f_{i_k}^{\varepsilon}\|_{\mathcal{C}(\mathcal{K})} \le 3\varepsilon \text{ for large } j, k.$$
(**)

• Suppose that (**) holds for the moment, we will now show how (*) can be obtained.

Pre-compactness criterion in C(K)

Proof

- (**) \Rightarrow (*): We will use a diagonal procedure.
 - * Using (**), take a subsequence $(f_{i_j}^1)$ of (f_i) such that $\|f_{i_j}^1 f_{i_k}^1\|_{\mathcal{C}(K)} \leq 1$ eventually.
 - * Now the sequence (f_{ij}¹) satisfies the condition of theorem. Since we are assuming (**), we can thus take a subsequence (f_{ij}²) of (f_{ij}¹) such that ||f_{ij}² f_{ik}²||_{C(K)} ≤ 1/2 eventually.
 * Proceeding inductively, we have a nest sequence of subsequences (f_i) ⊃ (f_{ij}¹) ⊃ (f_{ij}²) ⊃ ... such that, for each m ≥ 1,

$$\|f_{i_j}^m - f_{i_k}^m\|_{\mathcal{C}(\mathcal{K})} \leq 1/m$$
 eventually.

* Now let $f_{i_j} = f_{i_j}^j$. Then, for every fixed m, the sequence (f_{i_j}) is eventually a subsequence of $(f_{i_j}^m)$ and so $||f_{i_j} - f_{i_k}||_{C(K)} \le 1/m$ eventually. So (f_{i_j}) satisfies (*).

Pre-compactness criterion in C(K)

Proof

• We now prove (**), i.e. for every given ε , there is a subsequence $(f_{i_i}^{\varepsilon})$ of (f_i) such that

$$\|f_{i_j}^{\varepsilon} - f_{i_k}^{\varepsilon}\|_{\mathcal{C}(\mathcal{K})} \leq 3\varepsilon$$
 for large j, k .

- ★ By equi-continuity, there exists $\delta > 0$ such that $|f_i(x) - f_i(y)| < \varepsilon$ for all *i* and all $x, y \in K$ with $|x - y| < \delta$.
- * As K is compact, we can cover K by finitely many open balls $B(x_1, \delta), \ldots, B(x_N, \delta)$ with x_ℓ 's in K.
- * By uniform boundedness, for each ℓ , the sequence $(f_i(x_\ell))$ is bounded in \mathbb{R} . By Bolzano-Weierstrass' theorem, we can select a subsequence $(f_{i_i}^{\varepsilon})$ such that $(f_{i_i}^{\varepsilon}(x_\ell))$ is convergent for all ℓ . So

$$|f_{i_j}^{\varepsilon}(\mathsf{x}_{\ell}) - f_{i_k}^{\varepsilon}(\mathsf{x}_{\ell})| \leq \varepsilon \text{ for all } \ell \text{ and for all large } j, k.$$

- We now prove (**).
 - ★ ... $|f_i(x) f_i(y)| \le \varepsilon$ for all *i* and all *x*, *y* ∈ *K* with $|x y| < \delta$. ★ $B(x_1, \delta), \ldots, B(x_N, \delta)$ covers *K*. ★ ... $|f_{i_j}^{\varepsilon}(x_{\ell}) - f_{i_k}^{\varepsilon}(x_{\ell})| \le \varepsilon$ for all *ℓ* and for all large *j* and *k*.
 - * Now if $x \in K$, then $x \in B(x_{\ell}, \delta)$ for some ℓ . Then, for large j, k,

$$egin{aligned} |f_{i_j}^arepsilon(\mathbf{x}) - f_{i_k}^arepsilon(\mathbf{x}_\ell)| &\leq |f_{i_j}^arepsilon(\mathbf{x}_\ell) - f_{i_k}^arepsilon(\mathbf{x}_\ell)| + |f_{i_k}^arepsilon(\mathbf{x}_\ell) - f_{i_k}^arepsilon(\mathbf{x})| & \ &\leq 3arepsilon. \end{aligned}$$

* So
$$\|f_{i_j}^{\varepsilon} - f_{i_k}^{\varepsilon}\|_{\mathcal{C}(\mathcal{K})} \leq 3\varepsilon$$
, which proves (**).

Theorem (Komolgorov-Riesz-Fréchet's theorem)

Let $1 \leq p < \infty$ and Ω be an open bounded subset of \mathbb{R}^n . Suppose that a sequence (f_i) of $L^p(\Omega)$ satisfies

- (Boundedness) $\sup_i \|f_i\|_{L^p(\Omega)} < \infty$,
- (Equi-continuity in L^p) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\tau_y \tilde{f}_i \tilde{f}_i\|_{L^p(\Omega)} < \varepsilon$ for all $|y| < \delta$, where \tilde{f}_i is the extension by zero of f_i to the whole of \mathbb{R}^n .

Then, there exists a subsequence (f_{i_i}) which converges in $L^p(\Omega)$.

By definition $\tilde{f}_i : \mathbb{R}^n \to \mathbb{R}$ is given by $\tilde{f}_i = f_i$ in Ω and $\tilde{f}_i = 0$ in $\mathbb{R}^n \setminus \Omega$.

Pre-compactness criterion in $L^p(\Omega)$

Proof

 As in the proof of Ascoli-Arzelà's theorem, it suffices to show that, for every given ε > 0, there exists a subsequence (f^ε_{ij}) of (f_i) such that

$$\|f_{i_j}^{arepsilon} - f_{i_k}^{arepsilon}\|_{L^p(\Omega)} \leq 3arepsilon$$
 for large $j, k.$ (***)

- Claim: For every fixed $\varphi \in C_c^{\infty}(\mathbb{R}^n)$, the sequence $(\tilde{f}_i * \varphi|_{\bar{\Omega}})$ satisfies the condition of Ascoli-Arzelà's theorem.
 - * First, by Hölder's inequality, we have

$$\|\tilde{f}_i\|_{L^1(\mathbb{R}^n)} = \|f_i\|_{L^1(\Omega)} \le \|f_i\|_{L^p(\Omega)} |\Omega|^{1/p'}.$$

Thus, by the boundedness of (f_i) in $L^p(\Omega)$, we have that (\tilde{f}_i) is bounded in $L^1(\mathbb{R}^n)$.

Pre-compactness criterion in $L^p(\Omega)$

Proof

• Claim: $(\tilde{f}_i * \varphi|_{\bar{\Omega}})$ satisfies the condition of Ascoli-Arzelà's theorem.

*
$$\sup_{i} \|\tilde{f}_{i}\|_{L^{1}(\mathbb{R}^{n})} < \infty.$$

+ By Young's convolution ine

⋆ By Young's convolution inequality

$$\|\tilde{f}_i * \varphi\|_{L^{\infty}(\mathbb{R}^n)} \le \|\tilde{f}_i\|_{L^1(\mathbb{R}^n)} \|\varphi\|_{L^{\infty}(\mathbb{R}^n)}.$$

So
$$\sup_{i} \|\tilde{f}_{i} * \varphi\|_{\mathcal{C}(\bar{\Omega})} < \infty$$
.
Next,

$$egin{aligned} | ilde{f}_ist arphi(x)- ilde{f}_ist arphi(y)|&\leq \int_{\mathbb{R}^n} \left|arphi(x-z)-arphi(y-z)
ight|| ilde{f}(z)|\,dz\ &\leq \|arphi\|_{Lip(\mathbb{R}^n)}|x-y|\| ilde{f}_i\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

So by squeezing |x - y|, we can make $\sup_i |\tilde{f}_i * \varphi(x) - \tilde{f}_i * \varphi(y)|$ as small as we want.

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- $(\tilde{f}_i * \varphi|_{\bar{\Omega}})$ satisfies the condition of Ascoli-Arzelà's theorem.
- Now, take a non-negative function $\varrho \in C_c^{\infty}(B_1)$ with $\int_{\mathbb{R}^n} \varrho = 1$ and, for $\eta > 0$, let $\varrho_{\eta}(x) = \frac{1}{\eta^n} \varrho(x/\eta)$ be the standard mollifiers. Recall that we have the estimate

$$egin{aligned} \| ilde{f}_i st arrho_\eta - ilde{f}_i\|_{L^p}^p &\leq \int_{\mathbb{R}^n} |arrho_\eta(y)| \| au_{-y} ilde{f}_i - ilde{f}_i\|_{L^p}^p \, dy \ &\leq \sup_{|y|\leq \eta} \| au_y ilde{f}_i - ilde{f}_i\|_{L^p}^p \int_{\mathbb{R}^n} |arrho_\eta(y)| \, dy \ &= \sup_{|y|\leq \eta} \| au_y ilde{f}_i - ilde{f}_i\|_{L^p}^p. \end{aligned}$$

Pre-compactness criterion in $L^p(\Omega)$

Proof

- $(\tilde{f}_i * \varphi|_{\bar{\Omega}})$ satisfies the condition of Ascoli-Arzelà's theorem.
- $\|\tilde{f}_i * \varrho_\eta \tilde{f}_i\|_{L^p} \leq \sup_{|y| \leq \eta} \|\tau_y \tilde{f}_i \tilde{f}_i\|_{L^p}.$
- We are now ready to prove (***):
 - * By the equi-continuity, there exists a small $\eta > 0$ such that $\|\tilde{f}_i * \varrho_\eta \tilde{f}_i\|_{L^p} \leq \varepsilon$ for all *i*.
 - * Using Ascoli-Arzelà's theorem, select a subsequence $(f_{i_j}^{\varepsilon})$ of (f_i) such that $(\tilde{f}_{i_i}^{\varepsilon} * \varrho_{\eta}|_{\bar{\Omega}})$ is convergent in $C(\bar{\Omega})$.
 - * It follows that $\|\tilde{f}_{i_j}^{\varepsilon} * \varrho_{\eta} \tilde{f}_{i_k}^{\varepsilon} * \varrho_{\eta}\|_{L^p(\Omega)} \leq \varepsilon$ for large j, k.
 - * Consequently, by triangle inequality,

$$\begin{split} \|f_{i_{j}}^{\varepsilon} - f_{i_{k}}^{\varepsilon}\|_{L^{p}(\Omega)} &\leq \|f_{i_{j}}^{\varepsilon} \ast \varrho_{\eta} - f_{i_{k}}^{\varepsilon} \ast \varrho_{\eta}\|_{L^{p}(\Omega)} + \|f_{i_{j}}^{\varepsilon} \ast \varrho_{\eta} - f_{i_{j}}^{\varepsilon}\|_{L^{p}(\Omega)} \\ &+ \|f_{i_{k}}^{\varepsilon} \ast \varrho_{\eta} - f_{i_{k}}^{\varepsilon}\|_{L^{p}(\Omega)} \leq 3\varepsilon \text{ for large } j, k, \end{split}$$

which is (***).