C4.3 Functional Analytic Methods for PDEs
Lecture 5

Luc Nguyen
luc.nguyen@maths

University of Oxford

MT 2020

In the last 4 lectures

- L^p spaces and their properties.

This lecture

- Divergence theorem and Integration by parts formula.
- Weak derivatives.
- Sobolev spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ as Banach spaces.
- Differentiation rule for convolution of Sobolev functions.

Frequently used terminologies/notations

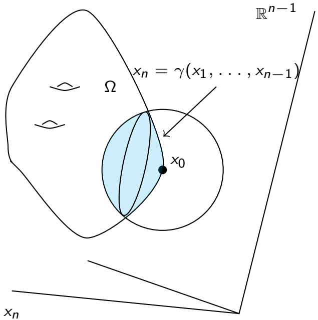
- Ω denotes a domain in $\mathbb{R}^n$.
- $C^k(\Omega)$ denotes the space of functions which are k-times continuously differentiable in Ω.
- $C^k(\bar{\Omega})$ denotes the subspace of $C^k(\Omega)$ consisting of functions which can be extended to a k-times continuously differentiable functions on some open set containing $\bar{\Omega}$.
- $C_c^k(\Omega)$ denotes the subspace of $C^k(\Omega)$ consisting of functions f such that $Supp(f) = \overline{\{f \neq 0\}}$ is a bounded closed subset of Ω.

Frequently used terminologies/notations

- Ω is said to be a Lipschitz (resp. C^k) domain, or equivalently, $\partial\Omega$ is said to be Lipschitz (resp. C^k), if for every $x_0 \in \partial\Omega$ there exists a radius $r_0 > 0$ such that, after a relabeling of coordinate axes if necessary,

$$\Omega \cap B_{r_0}(x_0) = \{x \in B_{r_0}(x_0) : x_n > \gamma(x_1, \ldots, x_{n-1})\}$$

for some Lipschitz (resp. C^k) function γ.



Divergence theorem

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Fact: $\partial\Omega$ admits an 'outward pointing' unit normal n.

Theorem (Divergence theorem)

Let $F \in C^1(\bar{\Omega}; \mathbb{R}^n)$. Then

$$\int_\Omega div\, F\, dx = \int_{\partial\Omega} F \cdot n\, dS.$$

In particular, if $F \in C_c^1(\Omega; \mathbb{R}^n)$, then

$$\int_\Omega div\, F\, dx = 0.$$

IBP formula

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.

Theorem (Integration by parts formula)

Let $f, g \in C^1(\bar{\Omega})$. Then

$$\int_\Omega f\,\partial_i g\,dx = \int_{\partial\Omega} f g n_i\,dS - \int_\Omega \partial_i f\,g\,dx.$$

In particular, if f or g has compact support in Ω, then

$$\int_\Omega f\,\partial_i g\,dx = -\int_\Omega \partial_i f\,g\,dx.$$

Weak derivatives

Let Ω be a domain in $\mathbb{R}^n$.

Definition

Let $f \in L^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index. A function $g \in L^1_{loc}(\Omega)$ is said to be a weak α-derivative of f if

$$\int_\Omega f\, \partial^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_\Omega g\varphi\, dx \text{ for all } \varphi \in C_c^\infty(\Omega). \tag{1}$$

We write $g = \partial^\alpha f$ in the weak sense.

The function φ is called a *test function*.

Example of weak derivatives

- If $f \in C^1(\Omega)$, then its classical derivatives are also its weak derivatives.
- Suppose $\Omega = (-1,1)$ and $f(x) = |x|$. Then, if $\varphi \in C_c^\infty(-1,1)$, we have by IBP that

$$\begin{aligned}\int_{-1}^{1} f(x)\,\varphi'(x)\,dx &= \int_{-1}^{0} (-x)\varphi'(x)\,dx + \int_0^1 x\varphi'(x)\,dx\\ &= -x\varphi(x)\Big|_{-1}^{0} - \int_{-1}^{0} (-1)\varphi(x)\,dx\\ &\qquad +x\varphi(x)\Big|_0^1 - \int_0^1 (1)\varphi(x)\,dx\\ &= -\int_{-1}^{1} \mathrm{sign}(x)\varphi(x)\,dx.\end{aligned}$$

So $f'(x) = \mathrm{sign}(x)$ in the weak sense.

Uniqueness of weak derivatives

Lemma

Let $f \in L^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index. The weak α-derivative of f, if exists, is uniquely defined up to a set of measure zero.

This follows from the definition of weak derivative and the following:

Lemma (Fundamental lemma of the Calculus of Variations)

Let $g \in L^1_{loc}(\Omega)$. If $\int_\Omega g\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, then $g = 0$ a.e. in Ω.

Uniqueness of weak derivatives

Proof

- We will only consider the case Ω is a bounded domain and $g \in L^1(\Omega)$. The general case is left as an exercise.
- In Sheet 1, you showed that $C_c^\infty(\Omega)$ is dense in $L^1(\Omega)$. Thus, for any $\varepsilon > 0$, we can select $h \in C_c^\infty(\Omega)$ such that $\|g - h\|_{L^1} \leq \varepsilon$. Furthermore, by triangle inequality $\|h\|_{L^1} \geq \|g\|_{L^1} - \varepsilon$.
- For $\delta > 0$, let $h_\delta = \frac{h}{\sqrt{\delta^2 + h^2}}$ so that $h_\delta \in C_c^\infty(\Omega)$ and $|h_\delta| \leq 1$.
- By hypotheses, $\int_\Omega g h_\delta \, dx = 0$.
- By construction, $\left| \int_\Omega (g - h) h_\delta \, dx \right| \leq \|g - h\|_{L^1} \|h_\delta\|_{L^\infty} \leq \varepsilon$.
- It follows that

$$\varepsilon \geq \int_\Omega g h_\delta \, dx - \int_\Omega (g - h) h_\delta \, dx = \int_\Omega h h_\delta \, dx.$$

Uniqueness of weak derivatives

Proof

- Recalling the expression of h_δ, we have

$$\varepsilon \geq \int_\Omega \frac{h^2}{\sqrt{\delta^2 + h^2}}\, dx.$$

- The integrand on the right hand side converges monotonically increasingly to $|h|$. Thus, by Lebesgue's monotone convergence theorem,

$$\varepsilon \geq \int_\Omega |h|\, dx = \|h\|_{L^1}.$$

- Recall that $\|h\|_{L^1} \geq \|g\|_{L^1} - \varepsilon$, we obtain that $2\varepsilon \geq \|g\|_{L^1}$. Sending $\varepsilon \to 0$, we obtain $\|g\|_{L^1} = 0$, i.e. $g = 0$ a.e. in Ω.

A relation between classical and weak derivatives

Remark

Suppose that

1. $f \in L^1(\Omega)$ *is weakly differentiable with weak derivatives* $\partial_1^w f$, $\ldots$, $\partial_n^w f$,
2. *and, for some subdomain* $\omega \subset \Omega$, f *is classically differentiable in* ω *with classical derivatives* $\partial_1^c f$, $\ldots$, $\partial_n^c f$.

Then

$$\partial_i^w f = \partial_i^c f \text{ a.e. in } \omega \text{ for all } i = 1, \ldots, n.$$

A relation between classical and weak derivatives

Sketch of proof

- Using the definition of weak derivatives, $f|_\omega$ is weakly differentiable with weak derivatives $\partial_1^w f|_\omega, \ldots, \partial_n^w f|_\omega$.
- As f is classically differentiable in ω, its classical derivatives are also weak derivatives of $f|_\omega$.
- By the uniqueness of weak derivatives, the conclusion follows.

Example of non-existence of weak derivatives

If $\Omega = (-1, 1)$ and $u(x) = \mathrm{sign}(x)$, then u has no weak derivative.
Proof

- Suppose otherwise that $u' = g \in L^1_{loc}(-1, 1)$. Then, for $\varphi \in C_c^\infty(-1, 1)$,

$$\begin{aligned}\int_{-1}^1 g(x)\varphi(x)\,dx &= \int_{-1}^0 \varphi'(x)\,dx - \int_0^1 \varphi'(x)\,dx\\ &= [\varphi(0) - \varphi(-1)] - [\varphi(1) - \varphi(0)]\\ &= 2\varphi(0).\end{aligned}$$

- In particular, if we take $\underline{\varphi \in C_c^\infty(-1, 0)}$, we have $\int_{-1}^0 g(x)\varphi(x)\,dx = 0$. So $g = 0$ a.e. in $(-1, 0)$. Likewise, $g = 0$ a.e. in $(0, 1)$. So $g = 0$ a.e. in $(-1, 1)$.
- We thus have $0 = \int_{-1}^1 g(x)\varphi(x)\,dx = 2\varphi(0)$ for all $\varphi \in C_c^\infty(-1, 1)$, which is impossible.

The Sobolev spaces $W^{k,p}(\Omega)$

- Ω: a domain of $\mathbb{R}^n$.
- For $k \geq 0$ and $1 \leq p \leq \infty$, define

$$W^{k,p}(\Omega) = \Big\{ f \in L^p(\Omega) \Big| \forall |\alpha| \leq k, \text{ the weak derivative } \partial^\alpha f \text{ exists and belongs to } L^p(\Omega) \Big\}.$$

  We equip $W^{k,p}(\Omega)$ with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \Big[\sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(\Omega)}^p \Big]^{\frac{1}{p}}$$

  so that $W^{k,p}(\Omega)$ is a normed vector space (check this!).
- For $p = 2$, we also write $H^k(\Omega)$ for $W^{k,2}(\Omega)$. These are inner product spaces (check this!) with inner product

$$\langle u, v \rangle_{W^{k,2}(\Omega)} = \sum_{|\alpha| \leq k} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(\Omega)}.$$

Examples of Sobolev functions

Let $\Omega = (-1, 1)$ and $f(x) = |x|$.

- We have that $f'(x) = \mathrm{sign}(x)$ and so $f \in W^{1,p}(-1, 1)$ for every $p \in [1, \infty]$.
- The function $f'(x) = \mathrm{sign}(x)$ has no weak derivatives, and so $f \notin W^{2,p}(-1, 1)$ for any $p \in [1, \infty]$.

Completeness of $W^{k,p}(\Omega)$

Theorem

For $k \geq 0$ and $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ is a Banach space. When $p = 2$, $W^{k,2}(\Omega)$ is a Hilbert space.

Proof

- We have seen that $W^{k,p}$ is a normed vector space and $W^{k,2}$ is an inner product space. It remains to show that $W^{k,p}$ is complete.
- Suppose that (u_m) is a Cauchy sequence in $W^{k,p}$. We need to show that there exists $u \in W^{k,p}$ such that $\|u_m - u\|_{W^{k,p}} \to 0$.
- For $|\alpha| \leq k$, $(\partial^\alpha u_m)$ is Cauchy in L^p, as

$$\|\partial^\alpha u_m - \partial^\alpha u_j\|_{L^p} \leq \|u_m - u_j\|_{W^{k,p}}.$$

  By Riesz-Fischer's theorem, we have that $(\partial^\alpha u_m)$ converges in L^p to some $v_\alpha \in L^p$.

Completeness of $W^{k,p}(\Omega)$

Proof

- (u_m) is Cauchy in $W^{k,p}$.
- For $|\alpha| \leq k$, $(\partial^\alpha u_m)$ converges in L^p to some $v_\alpha \in L^p$.
- To conclude, we show that $u := v_{(0,\ldots,0)}$ belongs to $W^{k,p}$ and $u_m \to u$ in $W^{k,p}$.
  - ⋆ By definition of weak derivatives, we have for $|\alpha| \leq k$ that

$$\int_\Omega u_m \partial^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega \partial^\alpha u_m \, \varphi \, dx \text{ for all } \varphi \in C_c^\infty(\Omega),$$

  - ⋆ Now we would like to pass $m \to \infty$. By Hölder's inequality

$$\Big| \int_\Omega (u_m - u) \partial^\alpha \varphi \, dx \Big| \leq \|u_m - u\|_{L^p} \|\partial^\alpha \varphi\|_{L^{p'}} \to 0.$$

    So $\int_\Omega u_m \partial^\alpha \varphi \, dx \to \int_\Omega u \partial^\alpha \varphi \, dx$.
  - ⋆ Similarly, $\int_\Omega \partial^\alpha u_m \, \varphi \, dx \to \int_\Omega v_\alpha \varphi \, dx$.

Proof

- (u_m) is Cauchy in $W^{k,p}$.
- For $|\alpha| \leq k$, $(\partial^\alpha u_m)$ converges in L^p to some $v_\alpha \in L^p$.
- $\star$ We thus have

$$\int_\Omega u \partial^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi \text{ for all } \varphi \in C_c^\infty(\Omega).$$

  So v_α is the weak α-derivative of u. So $u \in W^{k,p}$.

  $\star$ Now

$$\|u_m - u\|_{W^{k,p}}^p = \sum_{|\alpha| \leq k} \|\partial^\alpha u_m - \partial^\alpha u\|_{L^p}^p$$
$$= \sum_{|\alpha| \leq k} \|\partial^\alpha u_m - v_\alpha\|_{L^p}^p \stackrel{m\to\infty}{\longrightarrow} 0.$$

  So $u_m \to u$ in $W^{k,p}$.
- We conclude that $W^{k,p}$ is complete.

Reflexivity of $W^{k,p}(\Omega)$

Theorem

For $k \geq 0$ and $1 < p < \infty$, $W^{k,p}(\Omega)$ is reflexive.

Proof

- We will only consider the case $k = 1$. The general case requires some minor changes.
- By Eberlein's theorem, we only need to show that every bounded sequence in $W^{1,p}$ has a weakly convergent subsequence.
- Suppose $(u_m) \subset W^{1,p}$ is bounded. Then, (u_m) and $(\partial_i u_m)$ are bounded in L^p.
- By the weak sequential compactness property of L^p for $1 < p < \infty$, there exists a subsequence (u_{m_j}) such that (u_{m_j}) and $(\partial_i u_{m_j})$ are weakly convergent in L^p. Let u be the L^p weak limit of (u_{m_j}) and v_i be the L^p weak limit of $(\partial_i u_{m_j})$.

Reflexivity of $W^{k,p}(\Omega)$

- To conclude, we show that u belongs to $W^{1,p}$ and $u_{m_j} \rightharpoonup u$ in $W^{1,p}$.
- The proof that $u \in W^{1,p}$ is similar to the one we did moment ago, but also has some subtle difference: By definition of weak derivatives, we have

$$\int_\Omega u_{m_j} \partial_i \varphi = -\int_\Omega \partial_i u_{m_j}\, \varphi \text{ for all } \varphi \in C_c^\infty(\Omega),$$

  Sending $j \to \infty$ by using the definition weak convergence, we obtain

$$\int_\Omega u \partial_i \varphi = -\int_\Omega v_i\, \varphi \text{ for all } \varphi \in C_c^\infty(\Omega).$$

  So $v_i = \partial_i u$ in the weak sense. So $u \in W^{1,p}$.

Reflexivity of $W^{k,p}(\Omega)$

- It remains to show that, if $A \in (W^{1,p})^*$, then $Au_{m_j} \to Au$.
  - ⋆ Define $E : W^{1,p}(\Omega) \to (L^p(\Omega))^{n+1}$ by $Ef = (f, \partial_1 f, \ldots, \partial_n f)$. Then E is an isometry.
  - ⋆ Let $X := E(W^{1,p}(\Omega))$ and $Y := (L^p(\Omega))^{n+1}$. Define $\tilde{A} : X \to \mathbb{R}$ by $\tilde{A}p = AE^{-1}p$ for $p \in X$. Then $\tilde{A} \in X^*$. By Hahn-Banach's theorem, it has an extension $\hat{A} \in Y^*$.
  - ⋆ It follows that

$$\begin{aligned} Au_{m_j} = \tilde{A}Eu_{m_j} = \hat{A}Eu_{m_j} \\ &= \hat{A}(u_{m_j}, 0, \ldots, 0) + \sum_i \hat{A}(0, 0, \ldots, 0, \partial_i u_{m_j}, 0, \ldots, 0) \\ &=: B(u_{m_j}) + \sum_i B_i(\partial_i u_{m_j}) \\ &\to B(u) + \sum_i B_i(\partial_i u) = Au. \end{aligned}$$

This concludes the proof.

The Sobolev spaces $W_0^{k,p}(\Omega)$

- Ω: a domain of $\mathbb{R}^n$.
- For $k \geq 0$ and $1 \leq p < \infty$, define

$$W_0^{k,p}(\Omega) = \text{the closure of } C_c^\infty(\Omega) \text{ in } W^{k,p}(\Omega).$$

  When $p = 2$, we also write $H_0^k(\Omega)$ for $W_0^{k,2}(\Omega)$.
- In other words, $u \in W_0^{k,p}(\Omega)$ if there exist $u_m \in C_c^\infty(\Omega)$ such that $\|u_m - u\|_{W^{k,p}} \to 0$.
- When $k = 0$, $1 \leq p < \infty$, and Ω is a bounded domain, we have seen in Sheet 1 that $W_0^{0,p}(\Omega) = W^{0,p}(\Omega) = L^p(\Omega)$.
  In general, this is $\underline{\text{not true for } k \geq 1}$. Roughly speaking, $W_0^{k,p}(\Omega)$ consists of functions f in $W^{k,p}(\Omega)$ such that

$$\text{`}\partial^\alpha f = 0 \text{ on } \partial\Omega\text{' for all } |\alpha| \leq k - 1.$$

IBP formula for Sobolev functions

Proposition (Integration by parts)

Let $u \in W^{k,p}(\Omega)$ and $v \in W_0^{k,p'}(\Omega)$ with $k \geq 0$, $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\int_\Omega \partial^\alpha u v \, dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v \, dx \text{ for all } |\alpha| \leq k.$$

Proof

- By definition of $W_0^{k,p'}$, there exists $v_m \in C_c^\infty(\Omega)$ such that $v_m \to v$ in $W^{k,p'}$. In particular, $\partial^\alpha v_m \to \partial^\alpha v$ in $L^{p'}$ for all $|\alpha| \leq k$.
- By the definition of weak derivatives,

$$\int_\Omega \partial^\alpha u v_m \, dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v_m \, dx \text{ for all } |\alpha| \leq k.$$

IBP formula for Sobolev functions

Proof

- $\partial^\alpha v_m \to \partial^\alpha v$ in $L^{p'}$ for all $|\alpha| \leq k$.
- $\int_\Omega \partial^\alpha u v_m \, dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v_m \, dx$ for all $|\alpha| \leq k$.
- We can now pass $m \to \infty$ as in the proof of the completeness of Sobolev spaces.
  - ⋆ By Hölder's inequality

    $$\Big| \int_\Omega \partial^\alpha u (v_m - v) \, dx \Big| \leq \|\partial^\alpha u\|_{L^p} \|v_m - v\|_{L^{p'}} \to 0.$$

    So $\int_\Omega \partial^\alpha u v_m \, dx \to \int_\Omega \partial^\alpha u v \, dx$.
  - ⋆ Similarly, $\int_\Omega u \partial^\alpha v_m \, dx \to \int_\Omega u \partial^\alpha v \, dx$.
  - ⋆ We conclude that

    $$\int_\Omega \partial^\alpha u v \, dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v \, dx.$$

Differentiation rule for convolution of Sobolev functions

- Suppose $k \geq 0$ and $1 \leq p < \infty$.
- Let $f \in L^p(\mathbb{R}^n)$ and $g \in C^k_c(\mathbb{R}^n)$. We knew that $f * g \in C^k(\mathbb{R}^n)$ and
$$\partial^\alpha(f * g) = f * (\partial^\alpha g) \text{ for all } |\alpha| \leq k.$$

Lemma

Assume $f \in W^{k,p}(\mathbb{R}^n)$ and $g \in C^k_c(\mathbb{R}^n)$ for some $k \geq 0$ and $1 \leq p < \infty$, then

$$\partial^\alpha(f * g) = (\partial^\alpha f) * g \text{ for all } |\alpha| \leq k.$$

Differentiation rule for convolution of Sobolev functions

Proof

- We will only consider the case $k = 1$. We aim to prove that

$$\partial_{x_1}(f * g) = (\partial_{x_1} f) * g$$

- We compute

$$\begin{aligned}\partial_{x_1}(f * g)(x) = f * (\partial_{x_1} g)(x) &= \int_{\mathbb{R}^n} f(y)\, \partial_{x_1} g(x - y)\, dy \\ &= -\int_{\mathbb{R}^n} f(y)\, \partial_{y_1} g(x - y)\, dy \\ &= \int_{\mathbb{R}^n} \partial_{y_1} f(y)\, g(x - y)\, dy = ((\partial_{x_1} f) * g)(x).\end{aligned}$$

So we are done.