C4.3 Functional Analytic Methods for PDEs
Lecture 6

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MT 2020

In the last lecture

- Divergence theorem and Integration by parts formula.
- Definition of weak derivatives and
- Sobolev spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ as Banach spaces.
- Differentiation rule for convolution of Sobolev functions.

This lecture

- Density results for Sobolev spaces.
- Extension theorems for Sobolev functions.
- Trace (boundary value) of Sobolev functions.

Approximation of identity in Sobolev spaces

Theorem (Approximation of identity)

Let ϱ be a non-negative function in $C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varrho = 1$. For $\varepsilon > 0$, let

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n}\varrho\Big(\frac{x}{\varepsilon}\Big).$$

If $f \in W^{k,p}(\mathbb{R}^n)$ for some $k \geq 0$ and $1 \leq p < \infty$, then $f * \varrho_\varepsilon \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ and

$$\lim_{\varepsilon \to 0} \|f * \varrho_\varepsilon - f\|_{W^{k,p}(\mathbb{R}^n)} = 0.$$

In particular $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.

Approximation of identity in Sobolev spaces

Proof

- Let $f_\varepsilon = f * \varrho_\varepsilon$.
  - ⋆ As $\varrho_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, we have $f_\varepsilon \in C^\infty(\mathbb{R}^n)$.
  - ⋆ As $f \in L^p(\mathbb{R}^n)$ and $\varrho_\varepsilon \in L^1(\mathbb{R}^n)$, Young's inequality gives that $f_\varepsilon \in L^p(\mathbb{R}^n)$.
  - ⋆ The approximation of identity theorem in L^p gives that $\|f_\varepsilon - f\|_{L^p} \to 0$ as $\varepsilon \to 0$.
- By the differentiation rule for convolution of Sobolev functions, we have $\partial^\alpha f_\varepsilon = (\partial^\alpha f) * \varrho_\varepsilon$ for $|\alpha| \leq k$. Repeat the argument as above, we have $\partial^\alpha f_\varepsilon \in L^p(\mathbb{R}^n)$ and $\|\partial^\alpha f_\varepsilon - \partial^\alpha f\|_{L^p} \to 0$ as $\varepsilon \to 0$.
- We deduce that $f_\varepsilon \in W^{k,p}(\mathbb{R}^n)$ and

$$\|f_\varepsilon - f\|_{W^{k,p}} = \Big[\sum_{|\alpha| \leq k} \|\partial^\alpha f_\varepsilon - \partial^\alpha f\|_{L^p}^p\Big]^{1/p} \xrightarrow{\varepsilon \to 0} 0.$$

Meyers-Serrin's theorem

Theorem (Meyers-Serrin)

Suppose Ω is a domain in $\mathbb{R}^n$, $k \geq 0$ and $1 \leq p < \infty$. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$. Namely, for every $u \in W^{k,p}(\Omega)$ there exists a sequence $(u_m) \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that u_m converges to u in $W^{k,p}(\Omega)$.

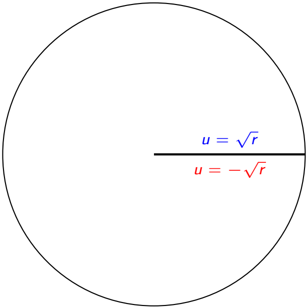
Remark: No regularity on Ω is assumed.

A question and an obstruction

Question

Is $C^\infty(\bar{\Omega}) \cap W^{k,p}(\Omega)$ dense in $W^{k,p}(\Omega)$?

Answer: Not always.

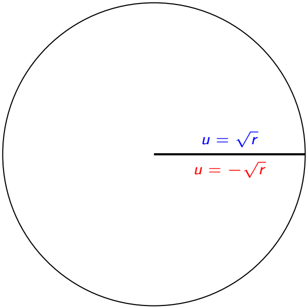


$\Omega = \{x^2 + y^2 < 1\} \setminus \{(x,0) | x \geq 0\}$
$\bar{\Omega} = \{x^2 + y^2 \leq 1\}$

Consider $u(x,y) = \sqrt{r} \cos \frac{\theta}{2}$ where $(x,y) = (r\cos\theta, r\sin\theta)$.
$u \in C^\infty(\Omega)$.
u is discontinuous in $\bar{\Omega}$.
One computes

$$\|u\|_{L^2}^2 = \int_\Omega u^2 \, dx \, dy = \int_0^1 \int_0^{2\pi} r \cos^2 \frac{\theta}{2} \, r \, dr \, d\theta = \frac{\pi}{3},$$

A question and an obstruction



$\Omega = \{x^2 + y^2 < 1\} \setminus \{(x,0) | x \geq 0\}$

$\bar{\Omega} = \{x^2 + y^2 \leq 1\}$

$D = \{x^2 + y^2 < 1\}$

Consider $u(x,y) = \sqrt{r}\cos\frac{\theta}{2}$.
$u \in C^\infty(\Omega)$ and $u \notin C(\bar{\Omega})$.
One computes $\|u\|_{L^2}^2 = \frac{\pi}{3}$,

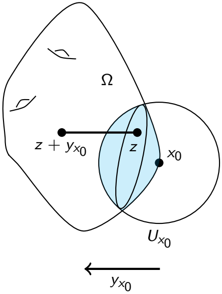
$$|\nabla u|^2 = (\partial_r u)^2 + \frac{1}{r^2}(\partial_\theta u)^2 = \frac{1}{4r},$$

$$\begin{aligned}\|\nabla u\|_{L^2}^2 &= \int_\Omega |\nabla u|^2\,dx\,dy \\ &= \int_0^1 \int_0^{2\pi} \frac{1}{4r}\,r\,dr\,d\theta = \frac{\pi}{2},\end{aligned}$$

So $u \in W^{1,2}(\Omega)$.
The jump discontinuity across $\theta = 0$ is an obstruction to approximate u by functions in $C^\infty(\bar{\Omega})$. It is in fact not possible, as $u \notin W^{1,2}(D)$.

The segment condition

- Ω: a domain in $\mathbb{R}^n$.
- Ω is said to satisfy the segment condition if every $x_0 \in \partial\Omega$ has a neighborhood U_{x_0} and a non-zero vector y_{x_0} such that if $z \in \bar{\Omega} \cap U_{x_0}$, then $z + ty_{x_0} \in \Omega$ for all $t \in (0,1)$.



- Note that if Ω is Lipschitz, then it satisfies the segment condition.

Approximation by functions in $C^\infty(\bar{\Omega})$

Theorem (Global approximation by functions smooth up to the boundary)

Suppose $k \geq 1$ and $1 \leq p < \infty$. If Ω satisfies the segment condition, then the set of restrictions to Ω of functions in $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\Omega)$. In particular $C^\infty(\bar{\Omega}) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

- An important consequence of the theorem is the statement that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ when $1 \leq p < \infty$. In order words $W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$.
- You will do the special when Ω is star-shaped in Sheet 2.

Extension by zero of functions in $W_0^{k,p}(\Omega)$

Lemma

Assume that $k \geq 0$ and $1 \leq p < \infty$. If $u \in W_0^{k,p}(\Omega)$, then its extension by zero $\bar{u}$ to $\mathbb{R}^n$ belongs to $W_0^{k,p}(\mathbb{R}^n)$.

Proof

- Suppose $u \in W_0^{k,p}(\Omega)$ and let $\bar{u}$ be its extension by zero to $\mathbb{R}^n$. It is tempted to say that, as $\bar{u} \equiv 0$ in $\mathbb{R}^n \setminus \Omega$,

$$\partial^\alpha \bar{u} = \begin{cases} \partial^\alpha u & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \qquad (*)$$

  which belongs to $L^p(\mathbb{R}^n)$, and call it the end of the proof. For this to work, we need to show first that $\bar{u}$ is weakly differentiable!

Extension by zero of functions in $W_0^{k,p}(\Omega)$

Proof

- Let $(u_m) \subset C_c^\infty(\Omega)$ be such that $u_m \to u$ in $W^{k,p}(\Omega)$. Let $\bar{u}_m$ be the extension by zero of u_m to $\mathbb{R}^n$. Then $\bar{u}_m \in C_c^\infty(\mathbb{R}^n)$ and

$$\|\bar{u}_m - \bar{u}_j\|_{W^{k,p}(\mathbb{R}^n)} = \|u_m - u_j\|_{W^{k,p}(\Omega)} \xrightarrow{m,j\to\infty} 0.$$

- We thus have that $(\bar{u}_m)$ is Cauchy in $W^{k,p}(\mathbb{R}^n)$ and thus converges in $W^{k,p}$ to some $\bar{u}_* \in W^{k,p}(\mathbb{R}^n)$.
- To conclude, we show that $\bar{u}_* = \bar{u}$ a.e. in $\mathbb{R}^n$.
  - ⋆ As $\bar{u}_m$ converges to $\bar{u}_*$ in $L^p(\mathbb{R}^n)$, there is a subsequence $\bar{u}_{m_j}$ which converges a.e. to $\bar{u}_*$ in $\mathbb{R}^n$. This implies that $\bar{u}_* = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$ and u_{m_j} converges a.e. to $\bar{u}_*$ in Ω.
  - ⋆ Likewise, as u_{m_j} converges to u in $L^p(\Omega)$, we can extract yet another subsequence $u_{m_{j_l}}$ which converges a.e. to u in Ω. It follows that $\bar{u}_* = u$ a.e. in Ω.
  - ⋆ So $\bar{u} = \bar{u}_*$ a.e. in $\mathbb{R}^n$.

More on extension

Theorem (Stein's extension theorem)

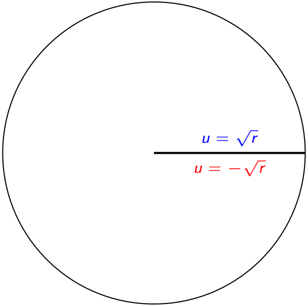
Assume that Ω is a bounded Lipschitz domain. Then there exists a linear operator E sending functions defined a.e. in Ω to functions defined a.e. in $\mathbb{R}^n$ such that for every $k \geq 0$, $1 \leq p < \infty$ and $u \in W^{k,p}(\Omega)$ it hold that $Eu = u$ a.e. in Ω and

$$\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C_{k,p,\Omega}\|u\|_{W^{k,p}(\Omega)}$$

The operator E is called a total extension for Ω.
You will have the opportunity to see how to construct such extension in a very specific case in Sheet 2.

More on extension

- There exists domain Ω for which there is no bounded linear operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$ such that $Eu = u$ a.e. in Ω.



$\Omega = \{x^2 + y^2 < 1\} \setminus \{(x,0) | x \geq 0\}$

$\bar{\Omega} = \{x^2 + y^2 \leq 1\}$

$D = \{x^2 + y^2 < 1\}$

We knew that the function $u(x,y) = \sqrt{r}\cos\frac{\theta}{2}$ satisfies

- $\star$ $u \in C^\infty(\Omega) \cap W^{1,2}(\Omega)$.
- $\star$ $u \notin W^{1,2}(D)$.

So no extension of u belongs to $W^{1,2}(\mathbb{R}^2)$.

Values of Sobolev functions on the boundary

- As prompted at the beginning of the course, in our later applications in the analysis of PDEs, solutions will live in a Sobolev space.
- When discussing PDEs on a domain, one needs to specify boundary conditions.
- A complication arises:
  - ⋆ On one hand, Sobolev 'functions' are equivalent classes of functions which are equal almost everywhere. Thus one can redefine the value of a Sobolev function on set of measure zero at will without changing the equivalent class it represents.
  - ⋆ On the other hand, the boundary of a domain usually has measure zero. So the boundary value of a Sobolev function cannot simply be defined by restricting as is the case for continuous functions.

Values of Sobolev functions on the boundary

Remark

Suppose $1 \leq p < \infty$, Ω *is a bounded smooth domain and let* $(X, \|\cdot\|)$ *be a normed vector space which contains* $C(\partial\Omega)$. *There is NO bounded linear operator* $T : L^p(\Omega) \to X$ *such that* $Tu = u|_{\partial\Omega}$ *for all* $u \in C(\bar{\Omega})$.

Proof

- Suppose by contradiction that such T exists. Consider $f_m \in C(\bar{\Omega})$ defined by

$$f_m(x) = \begin{cases} m\, dist(x, \partial\Omega) & \text{if } dist(x, \partial\Omega) < 1/m, \\ 1 & \text{if } dist(x, \partial\Omega) \geq 1/m. \end{cases}$$

- Then $\|f_m - 1\|_{L^p(\Omega)}^p \leq |\{dist(x, \partial\Omega) < 1/m\}| \leq \frac{C}{m}$ and so $f_m \to 1$ in $L^p(\Omega)$.
- Now as $Tf_m = 0 \not\to 1 = T1$ in X, T cannot be bounded.

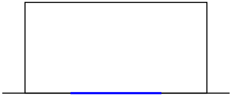
Values of Sobolev functions on the boundary

Theorem

Suppose $1 \le p < \infty$, *and that* Ω *is a bounded Lipschitz domain. Then there exists a* <u>*bounded*</u> *linear operator* $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, *called the trace operator, such that* $Tu = u|_{\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$.

We will only proof a weaker statement in a simpler situation:

$\Omega = \{x = (x', x_n) : |x'| < 2, \; 0 < x_n < 2\}$



$\hat{\Gamma} = \{x = (x', 0) : |x'| < 2\}$

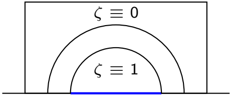
$\Gamma = \{x = (x', 0) : |x'| < 1\}$

We would like to define the trace operator relative to Γ: There exists a bounded linear operator $T_\Gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ such that

$$T_\Gamma u = u|_\Gamma \text{ for all } u \in C^1(\bar{\Omega}).$$

Values of Sobolev functions on the boundary

$\Omega = \{x = (x', x_n) : |x'| < 2, \; 0 < x_n < 2\}$



$\hat{\Gamma} = \{x = (x', 0) : |x'| < 2\}$

$\Gamma = \{x = (x', 0) : |x'| < 1\}$

$0 \leq \zeta \in C_c^\infty(B_{3/2})$ such that $\zeta \equiv 1$ in B_1

- We first prove the key estimate

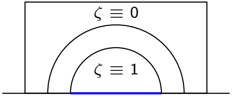
$$\|u\|_{L^p(\Gamma)} \leq C_p \|u\|_{W^{1,p}(\Omega)} \text{ for all } u \in C^1(\bar{\Omega}). \qquad (*)$$

  ⋆ We have

$$\int_\Gamma |u|^p \, dx' \leq \int_{\hat{\Gamma}} \zeta |u|^p \, dx' = -\int_{\hat{\Gamma}} \Big[\int_0^2 \partial_{x_n}(\zeta |u|^p) \, dx_n \Big] \, dx'$$
$$= -\int_\Omega \partial_{x_n}(\zeta |u|^p) \, dx \leq C_{p,\zeta} \int_\Omega [|u|^p + |Du||u|^{p-1}] \, dx.$$

Values of Sobolev functions on the boundary

$\Omega = \{x = (x', x_n) : |x'| < 2, \; 0 < x_n < 2\}$



$\hat{\Gamma} = \{x = (x', 0) : |x'| < 2\}$

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$\zeta \in C_c^\infty(B_{3/2})$ such that $\zeta \equiv 1$ in B_1.

- We first prove the key estimate

$$\|u\|_{L^p(\Gamma)} \leq C_p \|u\|_{W^{1,p}(\Omega)} \text{ for all } u \in C^1(\bar{\Omega}). \qquad (*)$$

  ⋆ We have $\displaystyle\int_\Gamma |u|^p \, dx' \leq C_{p,\zeta} \int_\Omega [|u|^p + |Du||u|^{p-1}] \, dx$.

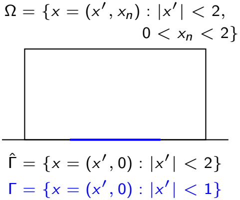
  ⋆ Using the inequality $|a||b|^{p-1} \leq \frac{1}{p}|a|^p + \frac{p-1}{p}|b|^p$,[1] we obtain

$$\int_\Gamma |u|^p \, dx' \leq C_{p,\zeta} \int_\Omega [|u|^p + |Du|^p] \, dx$$

  This proves (*).

[1] In the lecture, I said incorrectly that the last term was $|b|^{p'}$.

Values of Sobolev functions on the boundary



- We have proved the key estimate

$$\|u\|_{L^p(\Gamma)} \leq C_p \|u\|_{W^{1,p}(\Omega)} \text{ for all } u \in C^1(\bar{\Omega}). \qquad (*)$$

- It follows that the map $u \mapsto u|_\Gamma =: Au$ is a bounded linear operator from $(C^1(\bar{\Omega}), \|\cdot\|_{W^{1,p}})$ into $L^p(\Gamma)$.
- As Ω is Lipschitz, $C^\infty(\bar{\Omega})$ and hence $C^1(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$. Thus there exists a unique bounded linear operator $T_\Gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ which extends A, i.e. $T_\Gamma u = u|_\Gamma$ for all $u \in C^1(\bar{\Omega})$.

IBP formula revisited

Proposition (Integration by parts)

Suppose that $1 \le p < \infty$, Ω is a bounded Lipschitz domain, n be the outward unit normal to $\partial\Omega$, $T : W^{1,p}(\Omega) \to L^p(\Omega)$ is the trace operator, and $u \in W^{1,p}(\Omega)$. Then

$$\int_\Omega \partial_i u \, v \, dx = \int_{\partial\Omega} Tu \, v \, n_i \, dS - \int_\Omega u \, \partial_i v \, dx \text{ for all } v \in C^1(\bar{\Omega}).$$

Proof

- We knew that $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$. Thus there exists $u_m \in C^\infty(\bar{\Omega})$ such that $u_m \to u$ in $W^{1,p}$.
- Fix some $v \in C^1(\bar{\Omega})$. We have

$$\int_\Omega \partial_i u_m \, v \, dx = \int_{\partial\Omega} u_m \, v \, n_i \, dS - \int_\Omega u_m \, \partial_i v \, dx.$$

IBP formula revisited

Proof

- $\int_\Omega \partial_i u_m \, v \, dx = \int_{\partial\Omega} u_m \, v \, n_i \, dS - \int_\Omega u_m \, \partial_i v \, dx$.
- Note that $\partial_i u_m \to \partial_i u$, $u_m \to u$ in $L^p(\Omega)$ and $u_m|_{\partial\Omega} = Tu_m \to Tu$ in $L^p(\partial\Omega)$. We can thus argue using Hölder's inequality to send $m \to \infty$ to obtain

$$\int_\Omega \partial_i u \, v \, dx = \int_{\partial\Omega} Tu \, v \, n_i \, dS - \int_\Omega u \, \partial_i v \, dx$$

as wanted.

Functions of zero trace

Theorem (Trace-zero functions in $W^{1,p}$)

Suppose that $1 \leq p < \infty$, Ω is a bounded Lipschitz domain, $T : W^{1,p}(\Omega) \to L^p(\Omega)$ is the trace operator, and $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if $Tu = 0$.

Proof

- ($\Rightarrow$) Suppose $u \in W_0^{1,p}(\Omega)$. By definition, there exists $u_m \in C_c^\infty(\Omega)$ such that $u_m \to u$ in $W^{1,p}$. Clearly $Tu_m = 0$ and so by continuity, $Tu = 0$.
- ($\Leftarrow$) We will only consider the case Ω is the unit ball B. This proof can be generalised fairly quickly to star-shaped domains. The proof for Lipschitz domains is more challenging.

Functions of zero trace

Proof

- ($\Leftarrow$) Suppose that $u \in W^{1,p}(B)$ and $Tu = 0$. We would like to construct a sequence $u_m \in C_c^\infty(B)$ such that $u_m \to u$ in $W^{1,p}$.
  - ⋆ Let $\bar{u}$ be the extension by zero of u to $\mathbb{R}^n$.
  - ⋆ As $Tu = 0$, we have by the IBP formula that
  $$\int_B \partial_i u\, v\, dx = -\int_B u\, \partial_i v\, dx \text{ for all } v \in C^1(\bar{B}).$$
  It follows that
  $$\int_B \partial_i u\, v\, dx = -\int_B \bar{u}\, \partial_i v\, dx \text{ for all } v \in C_c^\infty(\mathbb{R}^n).$$
  By definition of weak derivatives, this means
  $$\partial_i \bar{u} = \begin{cases} \partial_i u & \text{in } B \\ 0 & \text{elsewhere} \end{cases} \quad \text{in the weak sense.}$$
  So $\bar{u} \in W^{1,p}(\mathbb{R}^n)$.

Functions of zero trace

Proof

- ($\Leftarrow$) We would like to construct a sequence $u_m \in C_c^\infty(B)$ such that $u_m \to u$ in $W^{1,p}(B)$.
  - Let $\bar{u}_\lambda(x) = \bar{u}(\lambda x)$. Observe that $Supp(\bar{u}_\lambda) \subset B_{1/\lambda}$.
  - In Sheet 1, you showed that $\bar{u}_\lambda \to \bar{u}$ in L^p as $\lambda \to 1$.
    Noting also that $\partial_i \bar{u}_\lambda(x) = \lambda \partial_i u(\lambda x)$, we also have that $\partial_i \bar{u}_\lambda \to \partial_i \bar{u}$ in L^p as $\lambda \to 1$.
    Hence $\bar{u}_\lambda \to \bar{u}$ in $W^{1,p}$ as $\lambda \to 1$.
  - Fix $\lambda_m > 1$ such that $\|\bar{u}_{\lambda_m} - \bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq 1/m$.
  - Let (ϱ_ε) be a family of mollifiers: $\varrho_\varepsilon(x) = \varepsilon^{-n}\varrho(x/\varepsilon)$ with $\varrho \in C_c^\infty(B)$, $\int_{\mathbb{R}^n} \varrho = 1$. Then $\bar{u}_{\lambda_m} * \varrho_\varepsilon \to \bar{u}_{\lambda_m}$ in $W^{1,p}$ as $\varepsilon \to 0$. Also, $Supp(\bar{u}_{\lambda_m} * \varrho_\varepsilon) \subset B_{\lambda_m^{-1}+\varepsilon}$. Thus, we can select ε_m sufficiently small such that $u_m := \bar{u}_{\lambda_m} * \varrho_{\varepsilon_m} \in C_c^\infty(B)$ and $\|u_m - \bar{u}_{\lambda_m}\|_{W^{1,p}(\mathbb{R}^n)} \leq 1/m$.
  - Now $\|u_m - u\|_{W^{1,p}(B)} \leq 2/m$ and so we are done.