C4.3 Functional Analytic Methods for PDEs
Lecture 7

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In the last 2 lectures

- Definition of Sobolev spaces
- Extension theorems for Sobolev functions.
- Trace (boundary value) of Sobolev functions.

This lecture

- Gagliardo-Nirenberg-Sobolev's inequality

Embeddings

Let X_1 and X_2 be two Banach spaces.

- We say X_1 is embedded in X_2 if $X_1 \subset X_2$.
- We say X_1 is <u>continuously</u> embedded in X_2 if X_1 is embedded in X_2 and the identity map $I : X_1 \to X_2$ is a <u>bounded</u> linear operator, i.e. there exists a constant C such that $\|x\|_{X_2} \leq C\|x\|_{X_1}$. We write $X_1 \hookrightarrow X_2$.
- We say X_1 is <u>compactly</u> embedded in X_2 if X_1 is embedded in X_2 and the identity map $I : X_1 \to X_2$ is a <u>compact bounded</u> linear operator. This means that I is continuous and every bounded sequence $(x_n) \subset X_1$ has a subsequence which is convergent with respect to the norm on X_2.

Our interest: The possibility of embedding $W^{k,p}$ in L^q or C^0.

Gagliardo-Nirenberg-Sobolev's inequality

Theorem (Gagliardo-Nirenberg-Sobolev's inequality)

Assume $1 \leq p < n$ and let $p^* = \frac{np}{n-p}$. Then there exists a constant $C_{n,p}$ such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C_{n,p} \|\nabla u\|_{L^p(\mathbb{R}^n)} \text{ for all } u \in W^{1,p}(\mathbb{R}^n).$$

In particular, $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$.

The number $p^* = \frac{np}{n-p}$ is called the Sobolev conjugate of p. It satisfies $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.
The case $p = 1$ is referred to as Gagliardo-Nirenberg's inequality.

GNS's inequality – Why $p < n$ and why p^*?

Question

For what p and q does it hold

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C_{n,p,q} \|\nabla u\|_{L^p(\mathbb{R}^n)} \text{ for all } u \in C_c^\infty(\mathbb{R}^n)? \qquad (*)$$

This will be answered by a scaling argument:

- Fix a non-zero function $u \in C_c^\infty(\mathbb{R}^n)$. Define $u_\lambda(x) = u(\lambda x)$. Then $u_\lambda \in C_c^\infty(\mathbb{R}^n)$ and so

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C_{n,p,q} \|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)}. \qquad (**)$$

- We compute

$$\|u_\lambda\|_{L^q}^q = \int_{\mathbb{R}^n} |u(\lambda x)|^q \, dx = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u(y)|^q \, dy = \lambda^{-n} \|u\|_{L^q}^q.$$

GNS's inequality – Why $p < n$ and why p^*?

- $u_\lambda(x) = u(\lambda x)$ and

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C_{n,p,q} \|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)}. \qquad (**)$$

- We compute $\|u_\lambda\|_{L^q} = \lambda^{-n/q} \|u\|_{L^q}$.
- Next,

$$\|\nabla u_\lambda\|_{L^p}^p = \int_{\mathbb{R}^n} |\lambda \nabla u(\lambda x)|^p \, dx$$

$$= \lambda^{p-n} \int_{\mathbb{R}^n} |\nabla u(y)|^p \, dy = \lambda^{p-n} \|\nabla u\|_{L^p}^p.$$

That is $\|\nabla u_\lambda\|_{L^p} = \lambda^{1-n/p} \|\nabla u\|_{L^p}$.

GNS's inequality – Why $p < n$ and why p^*?

- Putting in (**), we get

$$\lambda^{-n/q}\|u\|_{L^q} \leq C_{n,p,q}\lambda^{1-n/p}\|\nabla u\|_{L^p}.$$

Rearranging, we have

$$\lambda^{-1+\frac{n}{p}-\frac{n}{q}} \leq \frac{C_{n,p,q}\|\nabla u\|_{L^p}}{\|u\|_{L^q}}.$$

- Since the last inequality is valid for all λ, we must have that $-1 + \frac{n}{p} - \frac{n}{q} = 0$, i.e. $q = \frac{np}{n-p} = p^*$. As $q > 0$, we must also have $p \leq n$.
- We conclude that a necessary condition in order for the inequality (*) to hold is that $p \leq n$ and $q = p^*$.

GNS's inequality – Can $p = n$?

- Consider now the case $p = n$. Does it hold that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C_n \|\nabla u\|_{L^n(\mathbb{R}^n)} \text{ for all } u \in C_c^\infty(\mathbb{R}^n)? \qquad (\dagger)$$

  ⋆ When $n = 1$, this is true as

$$|u(x)| = \left| \int_{-\infty}^{x} u'(s)\,ds \right| \leq \int_{-\infty}^{\infty} |u'(s)|\,ds = \|u'\|_{L^1(\mathbb{R})}.$$

  ⋆ It turns out that $(\dagger)$ does not hold when $n \geq 2$. We will return to this after the proof of GNS's inequality.

Proof of GNS's inequality

- Recall that we would like to show, for $1 \leq p < n$ and $p^* = \frac{np}{n-p}$ that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C_{n,p}\|\nabla u\|_{L^p(\mathbb{R}^n)} \text{ for all } u \in W^{1,p}(\mathbb{R}^n). \qquad (\#)$$

- Claim 1: If $(\#)$ holds for functions in $C_c^\infty(\mathbb{R}^n)$, then it holds for functions in $W^{1,p}(\mathbb{R}^n)$.
  - ⋆ Take an arbitrary $u \in W^{1,p}(\mathbb{R}^n)$. As $p < \infty$, $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$. Hence, we can select $u_m \in C_c^\infty(\mathbb{R}^n)$ such that $u_m \to u$ in $W^{1,p}$.
  - ⋆ If $(\#)$ holds for functions in $C_c^\infty(\mathbb{R}^n)$, then $\|u_m\|_{L^{p^*}} \leq C_{n,p}\|\nabla u_m\|_{L^p}$.
  - ⋆ As $u_m \to u$ in $W^{1,p}$, we have $\partial_i u_m \to \partial_i u$ in L^p and so $\|\nabla u_m\|_{L^p} \to \|\nabla u\|_{L^p}$.
  - ⋆ Warning: It is tempted to try to show $\|u_m\|_{L^{p^*}} \to \|u\|_{L^{p^*}}$. However, this is false in general.

Proof of GNS's inequality

- Proof of Claim 1:
  - ⋆ $\|u_m\|_{L^{p^*}} \leq C_{n,p}\|\nabla u_m\|_{L^p}$.
  - ⋆ $\|\nabla u_m\|_{L^p} \to \|\nabla u\|_{L^p}$.
  - ⋆ As $u_m \to u$ in $W^{1,p}$, we have $u_m \to u$ in L^p, and so, we can extract a subsequence (u_{m_j}) which converges a.e. in $\mathbb{R}^n$ to u. By Fatou's lemma, we have

$$\int_{\mathbb{R}^n} |u|^{p^*}\,dx \leq \liminf_{j\to\infty} \int_{\mathbb{R}^n} |u_{m_j}|^{p^*}\,dx.$$

  - ⋆ So

$$\|u\|_{L^{p^*}} \leq \liminf_{j\to\infty} \|u_{m_j}\|_{L^{p^*}} \leq C_{n,p} \liminf_{j\to\infty} \|\nabla u_{m_j}\|_{L^p} = C_{n,p}\|\nabla u\|_{L^p}.$$

    So $(\#)$ holds.

Proof of GNS's inequality

- Claim 2: If (#) holds for $p = 1$, then it holds for all $1 < p < n$.
  - ⋆ Take an arbitrary non-trivial $u \in C_c^\infty(\mathbb{R}^n)$ and consider the function $v = |u|^\gamma$ with $\gamma > 1$ to be fixed. Clearly $v \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.
  - ⋆ In Sheet 3, you will show that $|u|$ is weakly differentiable and

$$\nabla|u| = \begin{cases} \nabla u & \text{in } \{x : u(x) > 0\}, \\ -\nabla u & \text{in } \{x : u(x) < 0\}, \\ 0 & \text{in } \{x : u(x) = 0\}. \end{cases}$$

  - ⋆ It follows that $\nabla v = \gamma |u|^{\gamma-1} \nabla |u| \in L^1(\mathbb{R}^n)$. So $v \in W^{1,1}(\mathbb{R}^n)$.
  - ⋆ Applying (#) in $W^{1,1}$ we get $\|v\|_{L^{\frac{n}{n-1}}} \leq C_n \|\nabla v\|_{L^1}$.
  - ⋆ On the left side, we have

$$\|v\|_{L^{\frac{n}{n-1}}} = \Big\{ \int_{\mathbb{R}^n} |v|^{\frac{n}{n-1}}\, dx \Big\}^{\frac{n-1}{n}} = \|u\|^\gamma_{L^{\frac{n\gamma}{n-1}}}.$$

Proof of GNS's inequality

- Claim 2: If (#) holds for $p = 1$, then it holds for all $1 < p < n$.
  - ⋆ $\|v\|_{L^{\frac{n}{n-1}}} \le C_n \|\nabla v\|_{L^1}$.
  - ⋆ On the left side, we have $\|v\|_{L^{\frac{n}{n-1}}} = \|u\|^{\gamma}_{L^{\frac{n\gamma}{n-1}}}$.
  - ⋆ On the right side, we use the inequality $|\nabla |u|| \le |\nabla u|$ and compute using Hölder's inequality:

$$\|\nabla v\|_{L^1} \le \int_{\mathbb{R}^n} \gamma |u|^{\gamma-1} |\nabla u|\,dx \le \gamma \Big\{ \int_{\mathbb{R}^n} |u|^{(\gamma-1)p'}\,dx \Big\}^{\frac{1}{p'}} \Big\{ \int_{\mathbb{R}^n} |\nabla u|^p\,dx \Big\}^{\frac{1}{p}}$$
$$= \gamma \|u\|^{\gamma-1}_{L^{(\gamma-1)p'}} \|\nabla u\|_{L^p}.$$

  - ⋆ Now we select γ such that $(\gamma-1)p' = \frac{n\gamma}{n-1}$, i.e. $\gamma = \frac{(n-1)p}{n-p}$ and obtain

$$\|u\|^{\gamma}_{L^{p^*}} \le C_n \gamma \|u\|^{\gamma-1}_{L^{p^*}} \|\nabla u\|_{L^p}.$$

    As $u \not\equiv 0$, we can divide both side by $\|u\|^{\gamma-1}_{L^{p^*}}$, and conclude Step 2.

Proof of GNS's inequality

- In view of Claim 1 and Claim 2, it thus remains to show GNS's inequality for smooth functions when $p = 1$. To better present the idea of the proof, I will only give the proof when $n = 2$, i.e.

$$\|u\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla u\|_{L^1(\mathbb{R}^2)} \text{ for all } u \in C_c^\infty(\mathbb{R}^2). \qquad (\Diamond)$$

(The case $n \geq 3$ can be dealt with in the same way (check this!).)

  - ⋆ The starting point is the argument we saw a bit earlier in the lecture. We have

$$|u(x)| = \Big|\int_{-\infty}^{x_1} \partial_{x_1} u(y_1, x_2)\,dy_1\Big| \leq \int_{-\infty}^{\infty} |\nabla u(y_1, x_2)|\,dy_1.$$

Likewise,

$$|u(x)| \leq \int_{-\infty}^{\infty} |\nabla u(x_1, y_2)|\,dy_2.$$

Proof of GNS's inequality

- We are proving

$$\|u\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla u\|_{L^1(\mathbb{R}^2)} \text{ for all } u \in C_c^\infty(\mathbb{R}^2). \qquad (\Diamond)$$

  - ⋆ We have $|u(x)| \leq \int_{-\infty}^{\infty} |\nabla u(y_1, x_2)|\, dy_1$ and $|u(x)| \leq \int_{-\infty}^{\infty} |\nabla u(x_1, y_2)|\, dy_2$.
  - ⋆ Multiplying the two inequalities gives

$$|u(x_1, x_2)|^2 \leq \Big\{ \int_{-\infty}^{\infty} |\nabla u(y_1, x_2)|\, dy_1 \Big\}\Big\{ \int_{-\infty}^{\infty} |\nabla u(x_1, y_2)|\, dy_2 \Big\}.$$

  - ⋆ Now note that the first integral on the right hand side is independent of x_1, and if one integrates the second integral on the right hand side with respect to x_1, one gets $\|\nabla u\|_{L^1}$. Thus, by integrating both side in x_1, we get

$$\int_{-\infty}^{\infty} |u(x_1, x_2)|^2\, dx_1 \leq \Big\{ \int_{-\infty}^{\infty} |\nabla u(y_1, x_2)|\, dy_1 \Big\} \|\nabla u\|_{L^1}.$$

Proof of GNS's inequality

- We are proving

$$\|u\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla u\|_{L^1(\mathbb{R}^2)} \text{ for all } u \in C_c^\infty(\mathbb{R}^2). \qquad (\diamondsuit)$$

  ⋆ We have shown

$$\int_{-\infty}^{\infty} |u(x_1, x_2)|^2 \, dx_1 \leq \Big\{ \int_{-\infty}^{\infty} |\nabla u(y_1, x_2)| \, dy_1 \Big\} \|\nabla u\|_{L^1}$$

  By the same line of argument, integrating the above in x_2 gives

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x_1, x_2)|^2 \, dx_1 \, dx_2 \leq \|\nabla u\|_{L^1}^2,$$

  which gives exactly $(\diamondsuit)$ with $C = 1$.

An improved Gagliardo-Nirenberg's inequality

Remark

By inspection, note that when $p = 1$, we actually prove the following slightly stronger inequality:

$$\|u\|^n_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \prod_{i=1}^n \|\partial_i u\|_{L^1(\mathbb{R}^n)}.$$

GNS's inequality for bounded domains

Theorem (Gagliardo-Nirenberg-Sobolev's inequality)

Assume that Ω is a bounded Lipschitz domain and $1 \le p < n$. Then, for every $q \in [1, p^*]$, there exists $C_{n,p,q,\Omega}$ such that

$$\|u\|_{L^q(\Omega)} \le C_{n,p,q,\Omega}\|u\|_{W^{1,p}(\Omega)} \text{ for all } u \in W^{1,p}(\Omega).$$

In particular, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$.

Proof

- Let $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ be an extension operator. Then

$$\|u\|_{L^{p*}(\Omega)} \le \|Eu\|_{L^{p*}(\mathbb{R}^n)} \le C_{n,p}\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C_{n,p}\|u\|_{W^{1,p}(\Omega)}.$$

- By Hölder inequality, we have $\|u\|_{L^q(\Omega)} \le \|u\|_{L^{p^*}(\Omega)}|\Omega|^{\frac{1}{q}-\frac{1}{p^*}}$.
- We conclude the proof with $C_{n,p,q,\Omega} = C_{n,p}|\Omega|^{\frac{1}{q}-\frac{1}{p^*}}$.

GNS's inequality – The case $p = n$ revisited

We proved earlier that the inequality

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C_n \|\nabla u\|_{L^n(\mathbb{R}^n)} \text{ for all } u \in C_c^\infty(\mathbb{R}^n) \qquad (\dagger)$$

is valid when $n = 1$ and mentioned that it is invalid when $n \geq 2$. Let us now prove the latter.

- We know that if ($\dagger$) holds then $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. Thus it suffices to exhibit a function $u \in W^{1,n}(\mathbb{R}^n) \setminus L^\infty(\mathbb{R}^n)$.
- It is enough to find $f \in W^{1,n}(B_2) \setminus L^\infty(B_1)$. The desired u then takes the form $u = f\zeta$ for any $\zeta \in C_c^\infty(B_2)$ with $\zeta \equiv 1$ in B_1.
- We impose that f is rotationally symmetric so that $f(x) = f(|x|) = f(r)$. Then we need to find a function $f : (0,2) \to \mathbb{R}$ such that

$$\int_0^2 [|f|^n + |f'|^n]\, r^{n-1}\, dr < \infty \text{ but } \operatorname*{ess\,sup}_{(0,1)} |f| = \infty.$$

GNS's inequality – The case $p = n$ revisited

- Then we need to find a function $f : (0,2) \to \mathbb{R}$ such that

$$\int_0^2 \left[|f|^n + |f'|^n\right] r^{n-1}\,dr < \infty \text{ but } \operatorname*{ess\,sup}_{(0,1)} |f| = \infty.$$

- The fact that $|f'|^n r^{n-1}$ is integrable implies that, near $r = 0$, f' is 'smaller' than $\frac{1}{r}$, so f is 'smaller' than $\ln r$.
- If we try $f = (\ln \frac{4}{r})^\alpha$, then $|f'|^n r^{n-1} = \frac{\alpha^n}{r} (\ln \frac{4}{r})^{n(\alpha-1)}$ is integrable for $\alpha \leq \frac{n-1}{n}$. Also, $|f|^n r^{n-1}$ is continuous in $[0,2]$ and hence integrable. So $f \in W^{1,n}(B_2)$ when $\alpha \leq \frac{n-1}{n}$.
- On the other hand, if $\alpha > 0$, then $\operatorname{ess\,sup}_{(0,1)} |f| = \infty$.

Trudinger's inequality

Theorem (Trudinger's inequality)

There exists a small constant $c_n > 0$ and a large constant $C_n > 0$ such that if $u \in W^{1,n}(\mathbb{R}^n)$, then $\exp\left[\left(\frac{c_n|u|}{\|u\|_{W^{1,n}(\mathbb{R}^n)}}\right)^{\frac{n}{n-1}}\right] \in L^1_{loc}(\mathbb{R}^n)$ *and*

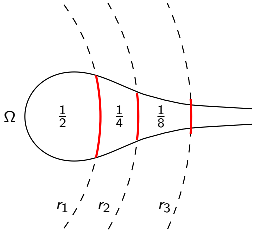
$$\sup_{x_0\in\mathbb{R}^n} \int_{B_1(x_0)} \exp\left[\left(\frac{c_n|u|}{\|u\|_{W^{1,n}(\mathbb{R}^n)}}\right)^{\frac{n}{n-1}}\right] dx \leq C_n.$$

A non-embedding theorem for unbounded domains

Theorem

Suppose $1 \le p < \infty$ *and* $\Omega \subset \mathbb{R}^n$ *be an unbounded domain with finite volume. Then* $W^{1,p}(\Omega)$ *does not embed into* $L^q(\Omega)$ *whenever* $q > p$.

Sketch of proof



- We may assume $|\Omega| = 1$. We need to construct a function $f \in W^{1,p}(\Omega) \setminus L^q(\Omega)$.
- Let $r_0 = 0$ and select r_k such that $\Omega_k := \Omega \cap \{r_k \le |x| < r_{k+1}\}$ has volume $\frac{1}{2^{k+1}}$.

Sketch of proof

- The function f will be of the form $f(x) = f(|x|)$ which is increasing in $|x|$. If we let $b_k = f(r_k)$, then

$$\|f\|_{L^p}^p = \sum_k \int_{\Omega_k} |f|^p \, dx \leq \sum_k b_{k+1}^p |\Omega_k| = \sum_k b_{k+1}^p 2^{-k-1}.$$

  Likewise, $\|f\|_{L^q}^q \geq \sum_k b_k^q 2^{-k-1}$.

- To make $\|f\|_{L^q} = \infty$, we then require that $b_k = 2^{k/q}$ infinitely many times.
  If we also impose that $b_k \leq 2^{k/q}$ for all k, then

$$\|f\|_{L^p}^p \leq \sum_k 2^{-k(1-\frac{p}{q})} < \infty.$$

A non-embedding theorem for unbounded domains

Sketch of proof

- $b_k = 2^{k/q}$ infinitely many times $\Rightarrow \|f\|_{L^q} = \infty$,
  $b_k \le 2^{k/q}$ for all $k \Rightarrow \|f\|_{L^p} < \infty$.
- Consider now $\|\nabla f\|_{L^p}$.
  - ⋆ On each Ω_k, we can arrange so that $|\nabla f| \sim \frac{b_{k+1}-b_k}{r_{k+1}-r_k}$.
  - ⋆ It is important to note that, for any fixed $\varepsilon > 0$, the inequality that $r_{k+1} - r_k > 2^{-\varepsilon k}$ must hold infinitely frequently. (As otherwise, $r_k \not\to \infty$.) Label them as $k_1 < k_2 < \dots$
  - ⋆ In Ω_{k_j}, we have $|\nabla f| \sim \frac{b_{k_j+1}-b_{k_j}}{r_{k_j+1}-r_{k_j}} \le 2^{k_j(1/q+\varepsilon)}$.
  - ⋆ In Ω_k with $k \neq k_j$, we control $|\nabla f|$ by imposing $b_{k+1} = b_k$ so that $|\nabla f| = 0$.
  - ⋆ To meet the requirement in the first bullet point, we ask $b_{k_j} = 2^{k_j/q}$.

A non-embedding theorem for unbounded domains

Sketch of proof

- $\|f\|_{L^q} = \infty$ and $\|f\|_{L^p} < \infty$.
- Consider $\|\nabla f\|_{L^p}$.
  - ⋆ Putting things together, we have

$$\|\nabla f\|_{L^p}^p = \sum_j \int_{\Omega_{k_j}} |\nabla f|^p \, dx$$
$$\leq \sum_j 2^{k_j(1/q+\varepsilon)p} 2^{-k_j-1} \leq \sum_j 2^{-k_j(1-\frac{p}{q}-\varepsilon p)}.$$

    Choosing $\varepsilon < \frac{1}{p} - \frac{1}{q}$, we see that this sum is finite.
- We conclude that $f \in W^{1,p}(\Omega)$ but $f \notin L^q(\Omega)$.