C4.3 Functional Analytic Methods for PDEs
Lecture 8

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In the last lecture

- Gagliardo-Nirenberg-Sobolev's inequality

This lecture

- Morrey's inequality

Hölder and Lipschitz continuity

- Let D be a subset of $\mathbb{R}^n$.
- For $\alpha \in (0, 1]$, we say that a function $u : D \to \mathbb{R}$ is (uniformly) α-Hölder continuous in D if there exists $C \geq 0$ such that

$$|u(x) - u(y)| \leq C|x - y|^\alpha \text{ for all } x, y \in D.$$

  The set of all α-Hölder continuous functions in D is denoted as $C^{0,\alpha}(D)$.
- When $\alpha = 1$, we use the term 'Lipschitz continuity' instead of '1-Hölder continuity'.

Hölder and Lipschitz continuity

- Note that, in our notation, when Ω is a bounded domain, $C^{0,\alpha}(\Omega) = C^{0,\alpha}(\bar{\Omega})$.
  In some text $C^{0,\alpha}(\Omega)$ is used to denote the set of continuous functions in Ω which is α-Hölder continuous on every compact subsets of Ω. In this course, we will use instead $C^{0,\alpha}_{loc}(\Omega)$ to denote this latter set, if such occasion arises.

$C^{0,\alpha}(D)$ is a Banach space

- For $u \in C^{0,\alpha}(D)$, let

$$[u]_{C^{0,\alpha}(D)} := \sup_{x,y\in D, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty.$$

and

$$\|u\|_{C^{0,\alpha}(D)} := \sup_D |u| + [u]_{C^{0,\alpha}(D)}.$$

Proposition

Let D be a subset of $\mathbb{R}^n$. Then $(C^{0,\alpha}(D), \|\cdot\|_{C^{0,\alpha}(D)})$ is a Banach space.

Hölder and Lipschitz continuity

Sketch of proof

- Piece 1: $\|\cdot\|_{C^{0,\alpha}(D)}$ is a norm.
  - ⋆ We will only give a proof for the statement that $[\cdot]_{C^{0,\alpha}(D)}$ satisfies the triangle inequality (i.e. it is a semi-norm). The rest is left as an exercise.
  - ⋆ Take $u, v \in C^{0,\alpha}(D)$. We want to show that $[u+v]_{C^{0,\alpha}(D)} \leq a + b$ where $a = [u]_{C^{0,\alpha}(D)}$ and $b = [v]_{C^{0,\alpha}(D)}$.
  - ⋆ Indeed, for any $x \neq y \in D$, we have $|u(x) - u(y)| \leq a|x-y|^\alpha$ and $|v(x) - v(y)| \leq b|x-y|^\alpha$. It follows that

$$|(u+v)(x) - (u+v)(y)| \leq (a+b)|x-y|^\alpha.$$

Divide both sides by $|x-y|^\alpha$ and take supremum we get

$$[u+v]_{C^{0,\alpha}(D)} = \sup_{x \neq y \in D} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq a + b,$$

as wanted.

$C^{0,\alpha}(D)$ is a Banach space

Sketch of proof

- Piece 2: $C^{0,\alpha}(D)$ is complete.
  - ⋆ Suppose that (u_m) is Cauchy in $C^{0,\alpha}(D)$.
  - ⋆ As $\|\cdot\|_{sup} \leq \|\cdot\|_{C^{0,\alpha}(D)}$, this implies that (u_m) is Cauchy in $C^0(\bar{D})$ and hence converges uniformly to some $u \in C^0(\bar{D})$.
  - ⋆ Claim: $u \in C^{0,\alpha}(D)$. Fix $\varepsilon > 0$. For every $x, y \in D$, we have

$$\begin{aligned} |(u_m - u_j)(x) - (u_m - u_j)(y)| &\leq [u_m - u_j]_{C^{0,\alpha}(D)}|x - y|^\alpha \\ &\leq \varepsilon|x - y|^\alpha \text{ for large } m, j. \end{aligned}$$

Sending $j \to \infty$, we thus have

$$|(u_m - u)(x) - (u_m - u)(y)| \leq \varepsilon|x - y|^\alpha \text{ for large } m.$$

Choose one such m we arrive at

$$|u(x) - u(y)| \leq \Big([u_m]_{C^{0,\alpha}(D)} + \varepsilon\Big)|x - y|^\alpha.$$

So $u \in C^{0,\alpha}(D)$.

$C^{0,\alpha}(D)$ is a Banach space

Sketch of proof

- Piece 2: $C^{0,\alpha}(D)$ is complete.
  - ⋆ Finally, we show that $u_m \to u$ in $C^{0,\alpha}(D)$. As u_m converges to u uniformly, it remains to show that $[u_m - u]_{C^{0,\alpha}(D)} \to 0$.
  - ⋆ Fix $\varepsilon > 0$. Recall from the previous slide that, for $x, y \in D$, we have

$$|(u_m - u)(x) - (u_m - u)(y)| \leq \varepsilon |x - y|^\alpha \text{ for large } m.$$

    Divide both sides by $|x - y|^\alpha$ and take supremum, we have

$$[u_m - u]_{C^{0,\alpha}(D)} \leq \varepsilon \text{ for large } m.$$

  - ⋆ As ε is arbitrary, we conclude that $[u_m - u]_{C^{0,\alpha}(D)} \to 0$.

An integral mean value inequality

Lemma

Let Ω be a domain in $\mathbb{R}^n$ and suppose $u \in C^1(\Omega)$. Then

$$\int_{B_r(x)} |u(y) - u(x)|dy \leq \frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u(y)|}{|y - x|^{n-1}}\,dy \text{ for all } B_r(x) \subset \Omega.$$

Proof

- It suffices to consider the case $x = 0$. We write $y = s\theta$ where $s \in [0, r)$ and $\theta \in \mathbb{S}^{n-1} \in \mathbb{R}^n$.
- By the fundamental theorem of calculus, we have

$$u(s\theta) - u(0) = \int_0^s \frac{d}{dt} u(t\theta)\,dt = \int_0^s \theta_i \partial_i u(t\theta)\,dt.$$

It follows that $|u(s\theta) - u(0)| \leq \int_0^s |\nabla u(t\theta)|\,dt.$

An integral mean value inequality

Proof

- $|u(s\theta) - u(0)| \leq \int_0^s |\nabla u(t\theta)|\, dt.$
- Integrating over θ and using Tonelli's theorem, we get

$$\begin{aligned}\int_{\partial B_1(0)} |u(s\theta) - u(0)|\, d\theta &\leq \int_0^s \int_{\partial B_1(0)} |\nabla u(t\theta)|\, d\theta\, dt \\ &= \int_0^s \int_{\partial B_t(0)} |\nabla u(y)|\, \frac{dS(y)}{t^{n-1}}\, dt \\ &= \int_{B_s(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy.\end{aligned}$$

An integral mean value inequality

Proof

- $\displaystyle\int_{\partial B_1(0)} |u(s\theta) - u(0)|\, d\theta \le \int_{B_s(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy.$
- Multiplying both sides by s^{n-1} and integrating over s, we get

$$\begin{aligned}\int_{B_r(0)} |u(y) - u(0)|\, dy &= \int_0^r \int_{\partial B_1(0)} |u(s\theta) - u(0)|\, d\theta s^{n-1} ds\\ &\le \int_{B_r(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy \int_0^r s^{n-1}\, ds\\ &= \frac{1}{n} r^n \int_{B_r(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy.\end{aligned}$$

This gives the desired integral mean value inequality.

A corollary of the integral mean value inequality

Corollary

Suppose $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$ *for some* $p > n$. *Then*

$$\int_{B_r(x)} |u(y) - u(x)|\, dy \leq C_{n,p} \|\nabla u\|_{L^p(B_r(x))} r^{\frac{n(p-1)}{p}+1} \text{ for all } B_r(x) \subset \Omega,$$

where the constant $C_{n,p}$ *depends only on* n *and* p.

Proof

- As in the previous proof, we assume without loss of generality that $x = 0$. We start with the integral mean value inequality:

$$\int_{B_r(0)} |u(y) - u(0)|\, dy \leq \frac{r^n}{n} \int_{B_r(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy.$$

A corollary of the integral mean value inequality

Proof

- By Hölder's inequality this gives

$$\int_{B_r(0)} |u(y) - u(0)|\,dy \leq \frac{r^n}{n} \|\nabla u\|_{L^p(B_r(0))} \Big\{ \int_{B_r(0)} \frac{1}{|y|^{(n-1)p'}}\,dy \Big\}^{1/p'}$$

$$= C_n r^n \|\nabla u\|_{L^p(B_r(0))} \Big\{ \int_0^r s^{-(n-1)(p'-1)}\,ds \Big\}^{1/p'}.$$

- As $p > n$, we have that $p' < \frac{n}{n-1}$ and so $(n-1)(p'-1) < 1$. Hence the integral in the curly braces converges to $C_{n,p} r^{-(n-1)(p'-1)+1}$. After a simplification, this gives

$$\int_{B_r(0)} |u(y) - u(0)|\,dy \leq C_{n,p} \|\nabla u\|_{L^p(B_r(0))} r^{\frac{n}{p'}+1},$$

which is the conclusion.

Morrey's inequality

Theorem (Morrey's inequality)

Assume that $n < p \leq \infty$. Then every $u \in W^{1,p}(\mathbb{R}^n)$ has a $(1 - \frac{n}{p})$-Hölder continuous representative. Furthermore there exists a constant $C_{n,p}$ such that

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u\|_{W^{1,p}(\mathbb{R}^n)}. \qquad (*)$$

In particular, $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.

Proof when $p < \infty$. The case $p = \infty$ will be dealt with in the next lecture.

- Step 1: Reduction to the case $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.
  - ⋆ Suppose that (*) holds for functions in $C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$. We show that this implies the theorem.

Morrey's inequality

Proof when $p < \infty$.

- Step 1: Reduction to the case $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.
  - ⋆ Let $u \in W^{1,p}(\mathbb{R}^n)$. As $p < \infty$, we can find $u_m \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$ such that $u_m \to u$ in $W^{1,p}$.
  - ⋆ Applying (*) to $u_m - u_j$ we have

$$\|u_m - u_j\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u_m - u_j\|_{W^{1,p}(\mathbb{R}^n)} \stackrel{m,j\to\infty}{\longrightarrow} 0.$$

    This means that (u_m) is Cauchy in $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$, and hence converges in $C^{0,1-\frac{n}{p}}$ to some $u_* \in C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.
  - ⋆ On the other hand, as $u_m \to u$ in L^p, a subsequence of it converges a.e. in $\mathbb{R}^n$ to u.
  - ⋆ It follows that $u = u_*$ a.e. in $\mathbb{R}^n$, i.e. u has a continuous representative.

Morrey's inequality

Proof when $p < \infty$.

- Step 1: Reduction to the case $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.
  - ⋆ We may thus assume henceforth that u is continuous, so that u_m converges to u in both $W^{1,p}$ and $C^{0,1-\frac{n}{p}}$.
  - ⋆ Now, applying (*) to u_m we have

$$\|u_m\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u_m\|_{W^{1,p}(\mathbb{R}^n)}.$$

Sending $m \to \infty$, we hence have

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

as wanted.

Morrey's inequality

Proof when $p < \infty$.

- Step 2: Proof of the C^0 bound in (*). We show that, for $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$, it holds that

$$|u(x)| \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \text{ for all } x \in \mathbb{R}^n. \qquad (**)$$

  - ⋆ By triangle inequality, we have

$$|B_1(x)||u(x)| \leq \int_{B_1(x)} |u(y) - u(x)|\,dy + \int_{B_1(x)} |u(y)|dy.$$

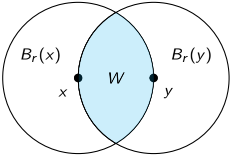
  - ⋆ By Hölder's inequality, the last integral is bounded by $C_{n,p}\|u\|_{L^p(B_1(x))}$.
  - ⋆ On the other hand, by the corollary to the integral mean value inequality, the first integral on the right hand side is bounded by $C_{n,p}\|\nabla u\|_{L^p(B_1(x))}$. The inequality (**) follows.

Morrey's inequality

Proof when $p < \infty$.

- Step 3: Proof of the $C^{0,1-\frac{n}{p}}$ semi-norm bound in (*). We show that, for $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$, it holds that

$$|u(x) - u(y)| \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}|x - y|^{1-\frac{n}{p}} \text{ for all } x, y \in \mathbb{R}^n. \quad (***)$$



- ⋆ If $x = y$, there is nothing to show. Suppose henceforth that $r = |x - y| > 0$ and let $W = B_r(x) \cap B_r(y)$.
- ⋆ Let a be the average of u in W, i.e. $a = \frac{1}{|W|}\int_W u(z)\,dz$. Then

$$|u(x) - u(y)| \leq |u(x) - a| + |u(y) - a|.$$

Morrey's inequality

Proof when $p < \infty$.

- Step 3: Proof of the $C^{0,1-\frac{n}{p}}$ semi-norm bound in (*).
  - ⋆ We estimate $|u(x) - a|$ as follows:

$$|u(x) - a| \leq \frac{1}{|W|} \int_W |u(x) - u(z)| dz$$
$$\leq \frac{C_n}{r^n} \int_{B_r(x)} |u(x) - u(z)| dz.$$

By the corollary to the mean value inequality, the right hand side is bounded by $C_{n,p} \|\nabla u\|_{L^p(B_r(x))} \, r^{1-\frac{n}{p}}$. So

$$|u(x) - a| \leq C_{n,p} \|\nabla u\|_{L^p(B_r(x))} \, r^{1-\frac{n}{p}}$$

  - ⋆ Similarly, $|u(y) - a| \leq C_{n,p} \|\nabla u\|_{L^p(B_r(y))} \, r^{1-\frac{n}{p}}$.
  - ⋆ Putting these together and recalling that $r = |x - y|$, we arrive at (***).

Morrey's inequality on domain for $n < p < \infty$

Theorem (Morrey's inequality)

Suppose that $n < p < \infty$ and Ω is a bounded Lipschitz domain. Then every $u \in W^{1,p}(\Omega)$ has a $(1 - \frac{n}{p})$-Hölder continuous representative and

$$\|u\|_{C^{0,1-\frac{n}{p}}(\Omega)} \leq C_{n,p,\Omega}\|u\|_{W^{1,p}(\Omega)}.$$

Indeed, let $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ be an extension operator. Then Eu has a continuous representative and

$$\begin{aligned}\|Eu\|_{C^{0,1-\frac{n}{p}}(\Omega)} &\leq \|Eu\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)}\\ &\leq C_{n,p}\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C_{n,p,\Omega}\|u\|_{W^{1,p}(\Omega)}.\end{aligned}$$

An improved integral mean value inequality

Lemma

Suppose $u \in C(\overline{B_R(0)}) \cap W^{1,p}(B_R(0))$ *for some* $p > n$. *Then, for every ball* $B_r(x) \subset \mathbb{R}^n$, *there holds*

$$\int_{B_r(x)} |u(y) - u(x)| dy \leq \frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u(y)|}{|y - x|^{n-1}} \, dy.$$

Proof

- Replacing p by any $\tilde{p} \in (n, p)$, we may assume that p is finite. Then we can find $u_m \in C^\infty(B_R(0)) \cap W^{1,p}(B_R(0))$ such that $u_m \to u$ in $W^{1,p}$. Furthermore, arguing as in Step 1 in the proof of Morrey's inequality, we also have that $u_m \to u$ in $C^{0,1-\frac{n}{p}}(\overline{B_R(0)})$.

An improved integral mean value inequality

Proof

- $u_m \to u$ in $W^{1,p}(B_R(0))$ and in $C^{0,1-\frac{n}{p}}(\overline{B_R(0)})$.
- By the integral mean value inequality for C^1 functions, we have

$$\int_{B_r(x)} |u_m(y) - u_m(x)|dy \leq \frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u_m(y)|}{|y-x|^{n-1}}\,dy.$$

- The left hand side converges to $\int_{B_r(x)} |u(y) - u(x)|dy$ since $u_m \to u$ uniformly.
- The right hand side converges to $\frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\,dy$ since $\nabla u_m \to \nabla u$ in L^p and since the function $y \mapsto \frac{1}{|y-x|^{n-1}}$ belongs to $L^{p'}$ (as noted in the proof of the corollary to the integral mean value inequality). The conclusion follows.