C4.3 Functional Analytic Methods for PDEs
Lecture 9

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In the last lecture

- Morrey's inequality, $n < p < \infty$

This lecture

- Morrey's inequality, $p = \infty$
- Friedrichs' inequality
- Rellich-Kondrachov's compactness theorem

Morrey's inequality

Theorem (Morrey's inequality)

Assume that $n < p \leq \infty$. *Then every* $u \in W^{1,p}(\mathbb{R}^n)$ *has a* $(1 - \frac{n}{p})$*-Hölder continuous representative. Furthermore there exists a constant* $C_{n,p}$ *such that*

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u\|_{W^{1,p}(\mathbb{R}^n)}. \tag{*}$$

In particular, $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.

Note that when $p = \infty$ we can no longer use the previous proof, as $C^\infty(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ is not dense in $W^{1,\infty}(\mathbb{R}^n)$.

Morrey's inequality

Proof when $p = \infty$.

- Suppose $u \in W^{1,\infty}(\mathbb{R}^n)$. Then $u \in W^{1,s}(B_R)$ for all $s < \infty$ and all ball B_R. By Morrey's inequality in the case of finite p, we thus have that u has a continuous representative, which we will assume to be u itself.
- By the improved integral mean value inequality, we have

$$\int_{B_r(x)} |u(y) - u(x)|dy \leq \frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u(y)|}{|y - x|^{n-1}} \, dy.$$

- Step 2 and Step 3 of the proof in the case $p < \infty$ can now be repeated to get

$$|u(x)| \leq C\|u\|_{W^{1,\infty}(\mathbb{R}^n)} \text{ for all } x \in \mathbb{R}^n. \qquad (**)$$

and

$$|u(x) - u(y)| \leq C\|u\|_{W^{1,\infty}(\mathbb{R}^n)}|x - y| \text{ for all } x, y \in \mathbb{R}^n. \qquad (***)$$

Morrey's inequality

Proof when $p = \infty$.

- It follows that

$$\|u\|_{C^{0,1}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,\infty}(\mathbb{R}^n)}$$

and we are done.

Morrey's inequality on domains

We make a couple of remarks:

- If Ω and p are such that there exists a bounded linear extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ (in particular $Eu = u$ a.e. in Ω for all $u \in W^{1,p}(\Omega)$), then

$$W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p}}(\Omega).$$

- The same proof on the whole space work on balls without establishing the existence of an extension operator. (Check this!)
- For general domains, one only has

$$W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p}}_{loc}(\Omega).$$

  (Revisit the example of the disk in $\mathbb{R}^2$ with a line segment removed.)

More on $W^{1,\infty}$

We have the following important theorem for the space $W^{1,\infty}(\Omega)$:

Theorem

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Then

$$W^{1,\infty}(\Omega) = C^{0,1}(\Omega).$$

Friedrichs' inequality

Theorem (Friedrichs' inequality)

Assume that Ω is a bounded open set and $1 \leq p < \infty$. Then, there exists $C_{p,\Omega}$ such that

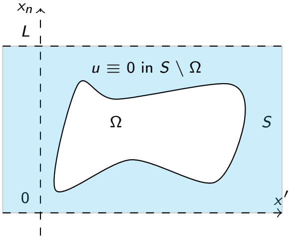
$$\|u\|_{L^p(\Omega)} \leq C_{p,\Omega} \|\nabla u\|_{L^p(\Omega)} \text{ for all } u \in W_0^{1,p}(\Omega).$$

Note that

- Only the derivatives of u appear on the right hand side.
- The function u belongs to $W_0^{1,p}(\Omega)$. The inequality is false for $u \in W^{1,p}(\Omega)$.
- By Friedrichs' inequality, when Ω is bounded, if we define $|||u||| = \|\nabla u\|_{L^p(\Omega)}$, then $||| \cdot |||$ is a norm on $W_0^{1,p}(\Omega)$ which is equivalent to the norm $\| \cdot \|_{W^{1,p}(\Omega)}$.
- In some text, Friedrichs' inequality is referred to as Poincaré's inequality.

Friedrichs' inequality

Proof



- We may assume that Ω is contain in the slab $S := \{(x', x_n) : 0 < x_n < L\}$.
- As usual, using the density of $C_c^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, it suffices to prove

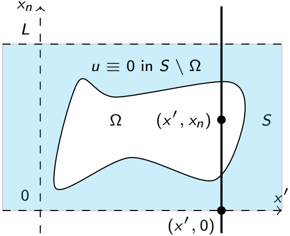
$$\|u\|_{L^p(\Omega)} \leq C_{p,\Omega}\|\nabla u\|_{L^p(\Omega)}$$

for $u \in C_c^\infty(\Omega)$.

- Take an arbitrary $u \in C_c^\infty(\Omega)$ and extend u by zero outside of Ω so that $u \in C_c^\infty(S)$.

Friedrichs' inequality

Proof



- Now, for every fixed x', we have

$$|u(x', x_n)| \leq \int_0^{x_n} |\partial_n u(x', t)| \, dt \leq \Big\{ \int_0^{x_n} |\partial_n u(x', t)|^p \, dt \Big\}^{1/p} x_n^{1/p'}$$
$$\leq \Big\{ \int_0^{L} |\partial_n u(x', t)|^p \, dt \Big\}^{1/p} x_n^{\frac{p-1}{p}}.$$

Friedrichs' inequality

Proof

- $|u(x', x_n)| \leq \left\{ \int_0^L |\partial_n u(x', t)|^p \, dt \right\}^{1/p} x_n^{\frac{p-1}{p}}.$
- It follows that

$$\int_0^L |u(x', x_n)|^p \, dx_n \leq \frac{1}{p} L^p \int_0^L |\partial_n u(x', t)|^p \, dt.$$

- Integrating over x' then gives

$$\begin{aligned} \|u\|_{L^p(\Omega)}^p &= \int_{\mathbb{R}^{n-1}} \int_0^L |u(x', x_n)|^p \, dx_n \, dx' \\ &\leq \frac{1}{p} L^p \int_{\mathbb{R}^{n-1}} \int_0^L |Du(x', t)|^p \, dt \, dx' = \frac{1}{p} L^p \|\nabla u\|_{L^p(\Omega)}^p. \end{aligned}$$

We are done.

Friedrichs-type inequality

Theorem (Friedrichs-type inequality)

Assume that Ω is a bounded open set and $1 \leq p < \infty$. Suppose that $1 \leq q \leq p^*$ if $p < n$, $1 \leq q < \infty$ if $p = n$, and $1 \leq q < \infty$ if $p > n$. Then there exists $C_{n,p,q,\Omega}$ such that

$$\|u\|_{L^q(\Omega)} \leq C_{n,p,q,\Omega} \|\nabla u\|_{L^p(\Omega)} \text{ for all } u \in W_0^{1,p}(\Omega).$$

Proof

- Extend u by zero to $\mathbb{R}^n$.
- If $p < n$, we have by Gagliardo-Nirenberg-Sobolev's inequality, that

$$\|u\|_{L^{p^*}(\Omega)} = \|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)} = C\|\nabla u\|_{L^p(\Omega)}.$$

  As Ω has finite measure, $\|u\|_{L^q(\Omega)} \leq C\|u\|_{L^{p^*}(\Omega)}$, and so we're done in this case.

Friedrichs-type inequality

Proof

- Note that, as Ω has finite measure, $W^{1,n}(\Omega) \hookrightarrow W^{1,\hat{p}}(\Omega)$ for any $\hat{p} < p$. The case $p = n$ thus follows from the previous case.
- When $p > n$, we have by Morrey's inequality that

$$\|u\|_{L^\infty(\Omega)} = \|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} = C\|u\|_{W^{1,p}(\Omega)}.$$

  By Friedrichs' inequality, we have $\|u\|_{W^{1,p}(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$. Also, as Ω has finite measure, $\|u\|_{L^q(\Omega)} \leq C\|u\|_{L^\infty(\Omega)}$. Putting these together we're also done in this case.

Rellich-Kondrachov's theorem

Theorem (Rellich-Kondrachov's compactness theorem)

Let Ω be a bounded Lipschitz domain and $1 \leq p \leq \infty$. Suppose $1 \leq q < p^*$ when $p < n$, $1 \leq q < \infty$ when $p = n$, and $1 \leq q \leq \infty$ when $p > n$. Then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, i.e. every bounded sequence in $W^{1,p}(\Omega)$ contains a subsequence which converges in $L^q(\Omega)$.

Critical embedding is not compact

Remark

For $1 \leq p < n$, the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is not compact.

Example by 'concentration'

- This example is by scaling. It is related to the argument in Lecture 7 to inspect for which p and q the space $W^{1,p}(\mathbb{R}^n)$ is embedded $L^q(\mathbb{R}^n)$.
- We may assume that the origin lies inside Ω and $B_{r_0} \subset \Omega$. Take an arbitrary non-zero function $u \in C_c^\infty(\mathbb{R}^n)$ with $Supp(u) \subset B_{r_0}$. We define, for $\lambda > 0$, $u_\lambda(x) = u(\lambda x)$.
- We knew that

$$\|u_\lambda\|_{L^q} = \lambda^{-n/q}\|u\|_{L^q} \text{ and } \|\nabla u_\lambda\|_{L^p} = \lambda^{1-n/p}\|\nabla u\|_{L^p}.$$

Critical embedding is not compact

Example by 'concentration'

- Hence, if we let $\hat{u}_\lambda = \lambda^{-1+n/p} u_\lambda$, then

$$\|\hat{u}_\lambda\|_{L^p} = \lambda^{-1}\|u\|_{L^p},$$
$$\|\hat{u}_\lambda\|_{L^{p^*}} = \|u\|_{L^{p^*}},$$
$$\|\nabla \hat{u}_\lambda\|_{L^p} = \|\nabla u\|_{L^p}.$$

  In particular, as $\lambda \to \infty$,

$$\|\hat{u}_\lambda\|_{W^{1,p}} \leq \|u\|_{W^{1,p}} \text{ and } \|\hat{u}_\lambda\|_{L^{p^*}} = \|u\|_{L^{p^*}} > 0.$$

Critical embedding is not compact

Example by 'concentration'

- Now if the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ was compact, then as $(\hat{u}_\lambda)$ is bounded in $W^{1,p}$, we could select a sequence $\lambda_k \to \infty$ such that $(\hat{u}_{\lambda_k})$ converges in $L^{p^*}(\Omega)$ to some limit $u_* \in L^{p^*}(\Omega)$.
- This would imply that

$$\|u_*\|_{L^{p^*}} = \lim_{k\to\infty} \|\hat{u}_{\lambda_k}\|_{L^{p^*}} = \|u\|_{L^{p^*}} > 0.$$

- On the other hand, $Supp(\hat{u}_\lambda) \subset B_{r_0/\lambda}$ and so $\hat{u}_\lambda \to 0$ a.e. in Ω as $\lambda \to \infty$. This would give that $u_* = 0$ a.e. which contradicts the above.

Critical embedding is not compact

Remark

For $1 \leq p < n$, the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ is not compact.

Example by 'translations'

- Take again an arbitrary non-zero function $u \in C_c^\infty(\mathbb{R}^n)$ and fix some unit vector e. Let $u_s(x) = u(x + se) = \tau_{se}u(x)$.
- Then $\|u_s\|_{W^{1,p}} = \|u\|_{W^{1,p}}$, $\|u_s\|_{L^{p^*}} = \|u\|_{L^{p^*}}$. Also $Supp(u_s) = \{x - se : x \in Supp(u)\}$ and so $u_s \to 0$ a.e. on $\mathbb{R}^n$ as $s \to \infty$.
- By the same reasoning, there is no sequence $s_k \to \infty$ such that u_{s_k} is convergent in L^{p^*}.

Pre-compactness criterion in $L^p(\Omega)$

Let us now do some preparation for the proof of Rellich-Kondrachov's theorem. Recall:

Theorem (Komolgorov-Riesz-Fréchet's theorem)

Let $1 \leq p < \infty$ and Ω be an open bounded subset of $\mathbb{R}^n$. Suppose that a sequence (f_i) of $L^p(\Omega)$ satisfies

1. *(Boundedness)* $\sup_i \|f_i\|_{L^p(\Omega)} < \infty$,
2. *(Equi-continuity in L^p) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\tau_y \tilde{f}_i - \tilde{f}_i\|_{L^p(\Omega)} < \varepsilon$ for all $|y| < \delta$, where $\tilde{f}_i$ is the extension by zero of f_i to all of $\mathbb{R}^n$.*

Then, there exists a subsequence (f_{i_j}) which converges in $L^p(\Omega)$.

In the case we are considering, boundedness follows from the embedding theorems. Let us now consider equi-continuity.

Continuity of translation operators in $W^{1,p}$

Lemma

Let $1 \leq p < \infty$. For every $v \in W^{1,p}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, it holds that

$$\|\tau_y v - v\|_{L^p(\mathbb{R}^n)} \leq |y| \|\nabla v\|_{L^p(\mathbb{R}^n)}.$$

Proof

- Using the density of $C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$ in $W^{1,p}(\mathbb{R}^n)$ for $p < \infty$, it suffices to consider $v \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.
- By the mean value theorem and Hölder's inequality, we have

$$\begin{aligned} |v(y+x) - v(x)| &\leq \int_0^1 |\frac{d}{dt} v(ty+x)|\, dt = \int_0^1 |y_i \partial_i v(ty+x)|\, dt \\ &\leq |y| \Big\{ \int_0^1 |\nabla v(ty+x)|^p\, dt \Big\}^{1/p}. \end{aligned}$$

Continuity of translation operators in $W^{1,p}$

Proof

- $|v(y+x) - v(x)|^p \leq |y|^p \int_0^1 |\nabla v(ty+x)|^p \, dt.$
- Integrating over x gives

$$\begin{aligned}
\|\tau_y v - v\|_{L^p}^p &= \int_{\mathbb{R}^n} |v(y+x) - v(x)|^p \, dx \\
&\leq |y|^p \int_{\mathbb{R}^n} \int_0^1 |\nabla v(ty+x)|^p \, dt \, dx \\
&= |y|^p \int_0^1 \int_{\mathbb{R}^n} |\nabla v(ty+x)|^p \, dx \, dt \\
&= |y|^p \|\nabla v\|_{L^p(\mathbb{R}^n)}^p.
\end{aligned}$$

So we have $\|\tau_y v - v\|_{L^p} \leq |y| \|\nabla v\|_{L^p(\mathbb{R}^n)}$ as wanted.

Continuity of translation operators in $W^{1,p}$

Remark

We remarked in Lecture 2 that the map $h \mapsto \tau_h$ is not a continuous map from $\mathbb{R}^n$ into $\mathscr{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$.
The above lemma implies that $h \mapsto \tau_h$ is not a continuous map from $\mathbb{R}^n$ into $\mathscr{L}(W^{1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))$.

Proof

- Let $X = \mathscr{L}(W^{1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))$. The statement amounts to $\tau_y \to Id$ in X as $y \to 0$. So we need to show that

$$0 = \lim_{y \to 0} \|\tau_y - Id\|_X = \lim_{y \to 0} \sup_{u \in W^{1,p}(\mathbb{R}^n) : \|u\|_{W^{1,p}} \leq 1} \|\tau_y u - u\|_{L^p}.$$

- By the lemma, we have $\|\tau_y u - u\|_{L^p} \leq |y| \|\nabla u\|_{L^p} \leq |y|$ whenever $\|u\|_{W^{1,p}} \leq 1$. So the point above is clear.

Characterisation of $W^{1,p}$ using translation operators

Theorem

Assume that $1 < p < \infty$ *and* $v \in L^p(\mathbb{R}^n)$*. Suppose that there exist small* $r > 0$ *and large* C *such that*

$$\|\tau_y v - v\|_{L^p(\mathbb{R}^n)} \leq C|y| \text{ for all } |y| \leq r.$$

Then

$$v \in W^{1,p}(\mathbb{R}^n) \text{ and } \|\nabla v\|_{L^p(\mathbb{R}^n)} \leq C.$$

Sketch of proof

- Fix a direction e_i. By hypothesis $q_t := \frac{1}{t}[\tau_{te_i} v - v]$ is bounded in L^p for $|t| \leq r$. By the weak sequential compactness property in L^p, we have along a sequence $t_k \to 0$ that q_{t_k} converges weakly in L^p to some $w_i \in L^p(\mathbb{R}^n)$.

Characterisation of $W^{1,p}$ using translation operators

Sketch of proof

- $q_{t_k} = \frac{1}{|t_k|}[\tau_{t_k e_i} v - v] \rightharpoonup w_i$ in L^p.
- The key point is the following identity

$$\int_{\mathbb{R}^n} [\tau_{t_k e_i} v - v]\varphi\, dx = -\int_{\mathbb{R}^n} v[\varphi - \tau_{-t_k e_i}\varphi]\, dx.$$

- Now divide both side by t_k and sending $k \to \infty$, we then get

$$\int_{\mathbb{R}^n} w_i \varphi\, dx = -\int_{\mathbb{R}^n} v \partial_i \varphi\, dx \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

This proves $\partial_i v = w_i \in L^p(\mathbb{R}^n)$. The conclusion follows.