C4.3 Functional Analytic Methods for PDEs
Lecture 10

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In the last lecture

- Morrey's inequality, $p = \infty$
- Friedrichs' inequality
- Statement of Rellich-Kondrachov's compactness theorem

This lecture

- Proof of Rellich-Kondrachov's compactness theorem
- Poincaré's inequality
- Local behavior of Sobolev functions

Rellich-Kondrachov's theorem

Theorem (Rellich-Kondrachov's compactness theorem)

Let Ω be a bounded Lipschitz domain and $1 \leq p \leq \infty$. Suppose $1 \leq q < p^*$ when $p < n$, $1 \leq q < \infty$ when $p = n$, and $1 \leq q \leq \infty$ when $p > n$. Then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, i.e. every bounded sequence in $W^{1,p}(\Omega)$ contains a subsequence which converges in $L^q(\Omega)$.

We reiterate that, when $p < n$, the endpoint embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is not compact.
When $p > n$, we have $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p}}(\Omega)$, so the above is a consequence of Ascoli-Arzelà's theorem. (Check this!)

Rellich-Kondrachov's theorem

Proof of the case $q = p \leq n$.

- Suppose that (u_m) is bounded in $W^{1,p}(\Omega)$. We need to construct a subsequence (u_{m_j}) which converges in $L^p(\Omega)$.
- As (u_m) is bounded in $L^p(\Omega)$, we would be done by Komolgorov-Riesz-Fréchet's theorem if (u_m) is equi-continuous in L^p sense.
- Last time we prove a continuity of translation operators in $W^{1,p}(\mathbb{R}^n)$. To make use of this result, we let $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ be a bounded linear extension operator. Then the family (Eu_m) is bounded in $L^p(\mathbb{R}^n)$ and is equi-continuous in $L^p(\mathbb{R}^n)$ sense. But as $\mathbb{R}^n$ is unbounded, we cannot apply Komolgorov-Riesz-Fréchet's theorem to this family.

Rellich-Kondrachov's theorem

Proof of the case $q = p \leq n$.

- We proceed as follows: Take a large ball B_R containing $\bar{\Omega}$ and select a cut-off function $\zeta \in C_c^\infty(B_R)$ such that $\zeta \equiv 1$ in Ω. Let

$$v_m = \zeta E u_m.$$

  Clearly $v_m = u_m$ a.e. in Ω, $Supp(v_m) \subset B_R$ and (v_m) is bounded in $W^{1,p}(\mathbb{R}^n)$.
- We aim to apply Komolgorov-Riesz-Fréchet's theorem to $(v_m|_{B_R})$.
  - ⋆ It is clear that $(v_m|_{B_R})$ is bounded in $L^p(B_R)$.
  - ⋆ Also, by the continuity of translation operators in $W^{1,p}$, we have

$$\|\tau_y v_m - v_m\|_{L^p(\mathbb{R}^n)} \leq |y| \|Dv_m\|_{L^p(\mathbb{R}^n)} \leq |y| \|v_m\|_{W^{1,p}(\mathbb{R}^n)}.$$

    Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\tau_y v_m - v_m\|_{L^p(B_R)} \leq \varepsilon$ for all m and all $|y| < \delta$, i.e. $(v_m|_{B_R})$ is equi-continuous in L^p sense. We're done.

Rellich-Kondrachov's theorem

Proof of the general case for $p \leq n$.

- Suppose that $1 \leq q < p^*$ if $p < n$, $1 \leq q < \infty$ if $p = n$. By the embedding theorems, we know that there exists $\hat{q} > q$ such that $W^{1,p}(\Omega) \hookrightarrow L^{\hat{q}}(\Omega)$.
- Suppose that (u_m) is bounded in $W^{1,p}(\Omega)$. We need to construct a subsequence (u_{m_j}) which converges in $L^q(\Omega)$.
- We knew from the previous case that there is a subsequence (u_{m_j}) which converges in $L^p(\Omega)$ to some $u \in L^p(\Omega)$. Passing to a subsequence if necessary, we may also assume that (u_{m_j}) converges to u a.e. in Ω.
- To conclude, we show that $u \in L^q(\Omega)$ and (u_{m_j}) converges in $L^q(\Omega)$ to u.
- If $q \leq p$, the above follows from Hölder's inequality. We assume henceforth that $q > p$.

Rellich-Kondrachov's theorem

Proof of the general case for $p \leq n$.

- We now show that $u \in L^q(\Omega)$. In fact, we show that $u \in L^{\hat{q}}(\Omega)$.
  - ⋆ By the embedding $W^{1,p}(\Omega) \hookrightarrow L^{\hat{q}}(\Omega)$, we have that u_m is bounded in $L^{\hat{q}}(\Omega)$.
  - ⋆ By Fatou's lemma, we have

$$\int_\Omega |u|^{\hat{q}}\,dx \leq \liminf_{j\to\infty} \int_\Omega |u_{m_j}|^{\hat{q}}\,dx < \infty.$$

Hence $u \in L^{\hat{q}}(\Omega)$.

Rellich-Kondrachov's theorem

Proof of the general case for $p \leq n$.

- Finally, we show that $u_{m_j} \to u$ in $L^q(\Omega)$.
  - We observe that $u_{m_j} - u$ converges to 0 in $L^p(\Omega)$ and is bounded in $L^{\hat{q}}(\Omega)$ with $p < q < \hat{q}$.
  - Now we write, for $\theta \in (0,1)$ to be fixed

$$\|u_{m_j} - u\|_{L^q}^q = \int_\Omega |u_{m_j} - u|^q \, dx = \int_\Omega |u_{m_j} - u|^{q\theta} |u_{m_j} - u|^{q(1-\theta)} \, dx$$

and apply Hölder's inequality with some pair of conjugate exponents r and r' to be fixed:

$$\|u_{m_j} - u\|_{L^q}^q \leq \Big\{ \int_\Omega |u_{m_j} - u|^{q\theta r} \, dx \Big\}^{1/r} \Big\{ \int_\Omega |u_{m_j} - u|^{q(1-\theta)r'} \, dx \Big\}^{1/r'}.$$

Rellich-Kondrachov's theorem

Proof of the general case for $p \leq n$.

- ...we are showing that $u_{m_j} \to u$ in $L^q(\Omega)$.
  - $u_{m_j} - u \to 0$ in $L^p(\Omega)$ and $u_{m_j} - u$ is bounded in $L^{\hat{q}}(\Omega)$ with $p < q < \hat{q}$.
  - $\|u_{m_j} - u\|_{L^q} \leq \|u_{m_j} - u\|_{L^{q\theta r}}^{\theta} \|u_{m_j} - u\|_{L^{q(1-\theta)r'}}^{1-\theta}$.
  - Now, if we can chose $\theta \in (0,1)$ and $r > 1$ such that $q\theta r = p$ and $q(1-\theta)r' = \hat{q}$, then the first factor on the right hand side goes to zero and the second factor remains bounded, and so $u_{m_j} \to u$ in $L^q(\Omega)$ as wanted.
  - To solve for θ and r, we first eliminate r to obtain

$$1 = \frac{1}{r} + \frac{1}{r'} = \theta \frac{p}{q} + (1-\theta)\frac{\hat{q}}{q}.$$

As $\frac{p}{q} < 1 < \frac{\hat{q}}{q}$, we can certainly select $\theta \in (0,1)$ satisfying the above. The exponent r is given by $r = \frac{q}{p\theta}$. This concludes the proof.

Poincaré's inequality

Theorem (Poincaré's inequality)

Suppose that $1 \le p \le \infty$ and Ω is a bounded Lipschitz domain. There exists a constant $C_{n,p,\Omega} > 0$ such that

$$\|u - \bar{u}_\Omega\|_{L^p(\Omega)} \le C_{n,p,\Omega} \|\nabla u\|_{L^p(\Omega)} \text{ for all } u \in W^{1,p}(\Omega),$$

where $\bar{u}_\Omega$ is the average of u in Ω:

$$\bar{u}_\Omega := \frac{1}{|\Omega|} \int_\Omega u(x)\,dx.$$

When $p = \infty$, the theorem is a consequence of the fact that $W^{1,\infty}(\Omega) = C^{0,1}(\Omega)$. (Check this!)

Poincaré's inequality

Proof for $p < \infty$.

- We argue by contradiction. Suppose the conclusion is not true. Then there exists a sequence $(u_m) \subset W^{1,p}(\Omega)$ such that

$$\|u_m - \bar{u}_m\|_{L^p} > m\|\nabla u_m\|_{L^p},$$

  where $\bar{u}_m$ is the average of u_m in Ω.
- Replacing u_m by $u_m - \bar{u}_m$, we may assume that u_m has zero average, so that $\|u_m\|_{L^p} > m\|\nabla u_m\|_{L^p}$.
- Replacing u_m by $\frac{1}{\|u_m\|_{L^p}} u_m$, we may assume that $\|u_m\|_{L^p} = 1$.
- The above implies that $\|\nabla u_m\|_{L^p} \leq \frac{1}{m}$ and so (u_m) is bounded in $W^{1,p}(\Omega)$.
- By Rellich-Kondrachov's compactness theorem, we can find a subsequence (u_{m_j}) which converges in $L^p(\Omega)$, say to u.

Poincaré's inequality

Proof for $p < \infty$.

- By the strong convergence of u_{m_j} to u, we have that
$$\|u\|_{L^p} = \lim_{j\to\infty} \|u_{m_j}\|_{L^p} = 1,$$
and
$$\int_\Omega u\,dx = \lim_{j\to\infty} \int_\Omega u_{m_j}\,dx = 0.$$
- On the other hand, as $\|\nabla u_m\|_{L^p} < \frac{1}{m}$, we have for every $\varphi \in C_c^\infty(\Omega)$ that
$$\int_\Omega u\partial_i\varphi\,dx = \lim_{j\to\infty} \int_\Omega u_{m_j}\partial_i\varphi\,dx = -\lim_{j\to\infty} \int_\Omega \partial_i u_{m_j}\varphi\,dx = 0.$$
Hence u is weakly differentiable and $\nabla u = 0$ in Ω. In Sheet 2, we show that this implies u is constant.
- As u has zero average, we must then have $u = 0$ in Ω, which contradicts the assertion that $\|u\|_{L^p} = 1$.

Local differentiability of Sobolev functions

Theorem

Suppose Ω is a domain in $\mathbb{R}^n$ and $n < p \leq \infty$. Assume that $u \in W^{1,p}(\Omega) \cap C(\Omega)$. Then u is differentiable a.e. in Ω and its derivatives equal its weak derivatives a.e. in Ω.

Proof

- We will only consider the case $p < \infty$. The case $p = \infty$ is a consequence.
- By Lebesgue's differentiation theorem, there is a set $Z \subset \Omega$ of measure zero such that

$$\lim_{r \to 0} \frac{1}{r^n} \int_{B_r(x)} |\nabla u(y) - \nabla u(x)|^p \, dy = 0 \text{ for all } x \in \Omega \setminus Z.$$

  We aim to show that u is differentiable at those $x \in \Omega \setminus Z$.

Local differentiability of Sobolev functions

Proof

- Fix some $x \in \Omega \setminus Z$ and consider the function

$$v(y) = u(y) - u(x) - \nabla u(x) \cdot (y - x) \text{ for } y \in \Omega.$$

  Then $v \in W^{1,p}(\Omega) \cap C(\Omega)$, $v(x) = 0$ and $\nabla v(y) = \nabla u(y) - \nabla u(x)$.

- By Morrey's inequality, we have for every ball $B_r(x) \Subset \Omega$ and $y \in \partial B_r(x)$ that

$$\begin{aligned} |v(y)| = |v(y) - v(x)| &\leq [v]_{C^{0,1-\frac{n}{p}}(B_r(x))} |x - y|^{1-\frac{n}{p}} \\ &\leq C r^{1-\frac{n}{p}} \|\nabla v\|_{L^p(B_r(x))} \\ &= C r^{1-\frac{n}{p}} \Big\{ \int_{B_r(x)} |\nabla u(y) - \nabla u(x)|^p \, dx \Big\}^{1/p}. \end{aligned}$$

Local differentiability of Sobolev functions

Proof

- So we have
  - ⋆ $\lim_{r\to 0} \frac{1}{r^n} \int_{B_r(x)} |\nabla u(y) - \nabla u(x)|^p \, dy = 0$, and
  - ⋆ $|v(y)| \leq C r^{1-\frac{n}{p}} \Big\{ \int_{B_r(x)} |\nabla u(y) - \nabla u(x)|^p \, dy \Big\}^{1/p}$ whenever $|y - x| = r$.

  Putting the two together, we see that

$$\lim_{y\to x} \frac{1}{|y-x|} |u(y) - u(x) - \nabla u(x) \cdot (y - x)| = \lim_{y\to x} \frac{1}{|y-x|} |v(y)| = 0.$$

  This means that u is differentiable at x and its classical gradient at x is the same at its weak gradient at x.

L^p differentiability of Sobolev functions

Theorem

Suppose Ω *is a domain in* $\mathbb{R}^n$ *and* $1 \leq p < n$*. Assume that* $u \in W^{1,p}(\Omega)$*. Then for almost all* $x \in \Omega$ *it holds that*

$$\lim_{r \to 0} \frac{1}{r^{1+\frac{n}{p}}} \Big\{ \int_{B_r(x)} |u(y) - u(x) - \nabla u(x) \cdot (y - x)|^p \, dy \Big\}^{1/p} = 0.$$

Discussion of proof

- As in the case $p > n$, we start by picking a set $Z \subset \Omega$ of measure zero such that

$$\lim_{r \to 0} \frac{1}{r^n} \int_{B_r(x)} |\nabla u(y) - \nabla u(x)|^p \, dy = 0 \text{ for all } x \in \Omega \setminus Z.$$

L^p differentiability of Sobolev functions

Discussion of proof

- We consider again the function

$$v(y) = u(y) - u(x) - \nabla u(x) \cdot (y - x) \text{ for } y \in \Omega,$$

so that $v \in W^{1,p}(\Omega)$ and $\nabla v(y) = \nabla u(y) - \nabla u(x)$. Note that however the meaning of $v(x) = 0$ is rather obscure since v does not have enough regularity.

- If we have the Poincaré-type inequality

$$\|v\|_{L^p(B_r(x))} \leq Cr\|\nabla v\|_{L^p(B_r(x))}, \qquad (*)$$

then, by recalling that $r^{-n}\|\nabla v\|^p_{L^p(B_r(x))} \to 0$ as $r \to 0$, we can obtain the conclusion as in the case $p > n$ considered previously. However, (*) is general not valid for arbitrary functions $v \in W^{1,p}$.

L^p differentiability of Sobolev functions

Discussion of proof

- The proof is actually much more involved and goes through approximation of u by smooth functions.
- It should be clear that the conclusion hold when $u \in C^1(\Omega)$ as

$$u(y) - u(x) - \nabla u(x) \cdot (y - x) = o(|y - x|) \text{ as } y \to x.$$

Towards L^p differentiability of Sobolev functions

We will be content with the following estimate for C^1 functions:

Lemma

Let Ω be a domain in $\mathbb{R}^n$ and suppose $u \in C^1(\Omega)$. Then, for all $B_r(x) \subset \Omega$,

$$\int_{B_r(x)} |u(y) - u(x) - \nabla u(x) \cdot (y - x)|^p \, dy \leq r^{p+n-1} \int_0^r \frac{1}{s^n} \int_{B_s(x)} |\nabla u(y) - \nabla u(x)|^p \, dy \, ds.$$

It should be noted that the term $\int_{B_s(x)} |\nabla u(y) - \nabla u(x)|^p \, dy$ is of order $o(s^n)$ as $u \in C^1$. This remains true for a.e. x if $u \in W^{1,p}$. Therefore, the right hand side is of order $o(r^{p+n})$. The theorem about L^p-differentiability thus makes sense.

Towards L^p differentiability of Sobolev functions

By letting $v(y) = u(y) - u(x) - \nabla u(x) \cdot (y - x)$, we obtain the following equivalent form:

Lemma

Let Ω be a domain in $\mathbb{R}^n$ and suppose $v \in C^1(\Omega)$. Then, for all $B_r(x) \subset \Omega$,

$$\int_{B_r(x)} |v(y) - v(x)|^p \, dy \leq r^{p+n-1} \int_0^r \frac{1}{s^n} \int_{B_s(x)} |\nabla v(y)|^p \, dy \, ds.$$

Proof

- By the converse to Hölder's inequality, we need to show that

$$\int_{B_r(x)} (v(y) - v(x)) g(y) \, dy \leq \left\{ r^{p+n-1} \int_0^r \frac{1}{s^n} \int_{B_s(x)} |\nabla v(y)|^p \, dy \, ds \right\}^{1/p}$$

for all $g \in L^{p'}(B_r(x))$ with $\|g\|_{L^{p'}} = 1$.

Towards L^p differentiability of Sobolev functions

Proof

- We may assume that $x = 0$. We have

$$v(y) - v(x) = \int_0^1 \frac{d}{dt} v(ty)\,dt = \int_0^1 y_i \partial_i v(ty)\,dt.$$

- Multiplying by $g(y)$ and integrating over $y \in \partial B_s(0)$ give

$$\int_{\partial B_s(0)} (v(y) - v(x)) g(y)\,dS(y) \leq \int_0^1 \int_{\partial B_s(0)} s|\nabla v(ty)||g(y)|dS(y)\,dt.$$

- Integrating over s then gives

$$\int_{B_r(0)} (v(y) - v(x)) g(y)\,dy \leq \int_0^r \int_0^1 \int_{\partial B_s(0)} s|\nabla v(ty)||g(y)|dS(y)\,dt\,ds.$$

Towards L^p differentiability of Sobolev functions

Proof

- Swapping the order of integration yields

$$\int_{B_r(0)} (v(y) - v(x))g(y)\,dy \leq r\int_0^1 \int_0^r \int_{\partial B_s(0)} |\nabla v(ty)||g(y)|dS(y)\,ds\,dt$$
$$= r\int_0^1 \int_{B_r(0)} |\nabla v(ty)||g(y)|dy\,dt.$$

- Using Hölder's inequality and note that $\|g\|_{L^{p'}} = 1$, we thus have

$$\int_{B_r(0))} (v(y) - v(x))g(y)\,dy \leq r\int_0^1 \Big\{\int_{B_r(0)} |\nabla v(ty)|^p\,dy\Big\}^{1/p}\,dt$$
$$\leq r\Big\{\int_0^1 \int_{B_r(0)} |\nabla v(ty)|^p\,dy\,dt\Big\}^{1/p}.$$

Proof

- It follows that

$$\int_{B_r(0))} (v(y) - v(x))g(y)\,dy \leq r\Big\{\int_0^1 \frac{1}{t^n}\int_{B_{tr}(0)} |\nabla v(z)|^p\,dz\,dt\Big\}^{1/p}$$
$$= r\Big\{r^{n-1}\int_0^r \frac{1}{s^n}\int_{B_s(0)} |\nabla v(z)|^p\,dz\,ds\Big\}^{1/p}.$$

As explained before, this together with the converse to Hölder's inequality gives the conclusion.