C4.3 Functional Analytic Methods for PDEs
Lecture 14

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In the last 3 lectures

- Existence of weak solutions to linear elliptic equations.

This lecture

- H^2 regularity of weak solutions to linear elliptic equations.

The question of regularity

In the rest of this course we consider regularity results for solutions to

$$Lu = -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu = f \text{ in a domain } \Omega$$

with $f \in L^2(\Omega)$.

- We want to keep in mind the following two motivating examples in $1d$:
$$-u'' = f \text{ in } (-1,1) \qquad (*)$$
and
$$-(au')' = f \text{ in } (-1,1) \text{ where } a = \chi_{(-1,0)} + 2\chi_{(0,1)}. \qquad (**)$$
- For (*), u belongs to H^2.
- For (**), au' belongs to H^1. Typically this implies u' is discontinuous and hence $u \notin H^2$. Nevertheless u is continuous.

Interior H^2 regularity

Theorem (Interior H^2 regularity)

Suppose that $a \in C^1(\Omega)$, $b, c \in L^\infty(\Omega)$, *a is uniformly elliptic, and* $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. *Suppose that* $f \in L^2(\Omega)$.
If $u \in H^1(\Omega)$ *satisfies* $Lu = f$ *in* Ω *in the weak sense then* $u \in H^2_{loc}(\Omega)$, *and for any open* ω *such that* $\bar{\omega} \subset \Omega$ *we have*

$$\|u\|_{H^2(\omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})$$

where the constant C *depends only on* $n, \Omega, \omega, a, b, c$.

Global H^2 regularity

Theorem (Global H^2 regularity)

Suppose that Ω is a bounded domain and $\partial\Omega$ is C^2 regular. Suppose that $a, b, c \in C^1(\bar{\Omega})$, a is uniformly elliptic, and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. Suppose that $f \in L^2(\Omega)$. If $u \in H_0^1(\Omega)$ satisfies $Lu = f$ in Ω in the weak sense then $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})$$

where the constant C depends only on n, Ω, a, b, c.

Remark: If $\partial\Omega$ is C^∞, $a, b, c \in C^\infty(\bar{\Omega})$, and $f \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$.

The case of $-\Delta$

To illustrate the idea, we focus in the case a is constant, $b \equiv 0$, $c \equiv 0$. The local H^2 regularity result is equivalent to:

Theorem (Interior H^2 regularity for $-\Delta$)

Suppose $f \in L^2(B_2)$ and $u \in H^1(B_2)$. If $-\Delta u = f$ in B_2 in the weak sense, then $u \in H^2(B_1)$ and

$$\|u\|_{H^2(B_1)} \leq C(\|f\|_{L^2(B_2)} + \|u\|_{H^1(B_2)})$$

where the constant C depends only on n.

The case of $-\Delta$

The start of the proof is the following simple but important lemma:

Lemma

Suppose that $u \in C_c^\infty(\mathbb{R}^n)$*. Then*

$$\|\nabla^2 u\|_{L^2(\mathbb{R}^n)} = \|\Delta u\|_{L^2(\mathbb{R}^n)}.$$

The proof is a computation using integration by parts:

$$\begin{aligned}\|\nabla^2 u\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \partial_i \partial_j u \partial_i \partial_j u \, dx = -\int_{\mathbb{R}^n} \partial_j u \partial_j \partial_i^2 u \, dx \\ &= \int_{\mathbb{R}^n} \partial_j^2 u \partial_i^2 u \, dx = \|\Delta u\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

The case of $-\Delta$

The following lemma is a generalisation in the weak setting:

Lemma

Suppose that $f \in L^2(\mathbb{R}^n)$, $u \in H^1(\mathbb{R}^n)$ and u has compact support. Suppose that $-\Delta u = f$ in $\mathbb{R}^n$ in the weak sense. Then $u \in H^2(\mathbb{R}^n)$ and

$$\|\nabla^2 u\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}.$$

Proof of the lemma

- Take a family of mollifiers (ϱ_ε): Fix a non-negative function $\varrho \in C_c^\infty(B_1)$ with $\int_{\mathbb{R}^n} \varrho = 1$ and let $\varrho_\varepsilon(x) = \varepsilon^{-n}\varrho(x/\varepsilon)$.
- Set $u_\varepsilon = \varrho_\varepsilon * u$ and $f_\varepsilon = \varrho_\varepsilon * f$.
  Then $u_\varepsilon, f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ and $u_\varepsilon \to u$ in $H^1(\mathbb{R}^n)$ and $f_\varepsilon \to f$ in $L^2(\mathbb{R}^n)$.

The case of $-\Delta$

Proof of the lemma

- Claim: $-\Delta u_\varepsilon = f_\varepsilon$ in $\mathbb{R}^n$.
  - ⋆ Fix $v \in C_c^\infty(\mathbb{R}^n)$ and consider $\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v \, dx$.
  - ⋆ Recall that, as $u \in H^1(\mathbb{R}^n)$, $\nabla u_\varepsilon = \varrho_\varepsilon * \nabla u$.
  - ⋆ Hence, by Fubini's theorem,

$$\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v \, dx = \int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y) \partial_{y_i} u(y) \, dy \Big] \partial_{x_i} v(x) \, dx$$
$$= \int_{\mathbb{R}^n} \partial_{y_i} u(y) \Big[\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y) \partial_{x_i} v(x) \, dx \Big] \, dy.$$

  - ⋆ Integrating by parts in the inner integral we get

$$\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v \, dx = -\int_{\mathbb{R}^n} \partial_{y_i} u(y) \Big[\int_{\mathbb{R}^n} \partial_{x_i} \varrho_\varepsilon(x-y) v(x) \, dx \Big] \, dy.$$

The case of $-\Delta$

Proof of the lemma

- Claim: $-\Delta u_\varepsilon = f_\varepsilon$ in $\mathbb{R}^n$.
  - ⋆ $\displaystyle\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\,dx = -\int_{\mathbb{R}^n} \partial_{y_i} u(y)\Big[\int_{\mathbb{R}^n} \partial_{x_i}\varrho_\varepsilon(x-y)v(x)\,dx\Big]\,dy.$
  - ⋆ Now observe that $\partial_{x_i}\varrho_\varepsilon(x-y) = -\partial_{y_i}\varrho_\varepsilon(x-y)$.
  - ⋆ We thus have, by Fubini's theorem again,

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\,dx &= \int_{\mathbb{R}^n} \partial_{y_i} u(y)\Big[\int_{\mathbb{R}^n} \partial_{y_i}\varrho_\varepsilon(x-y)v(x)\,dx\Big]\,dy\\ &= \int_{\mathbb{R}^n}\Big[\int_{\mathbb{R}^n} \partial_{y_i} u(y)\partial_{y_i}\varrho_\varepsilon(x-y)\,dy\Big]v(x)\,dx.\end{aligned}$$

  - ⋆ As $-\Delta u = f$ in the weak sense, the inner integral is equal to $\displaystyle\int_{\mathbb{R}^n} f(y)\,\varrho_\varepsilon(x-y)\,dy$, which is $f_\varepsilon(x)$.
  - ⋆ We deduce that

$$\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\,dx = \int_{\mathbb{R}^n} f_\varepsilon(x)v(x)\,dx.$$

The case of $-\Delta$

Proof of the lemma

- Claim: $-\Delta u_\varepsilon = f_\varepsilon$ in $\mathbb{R}^n$.
  - ⋆ As v was picked arbitrarily in $C_c^\infty(\mathbb{R}^n)$, we have that $-\Delta u_\varepsilon = f_\varepsilon$ in $\mathbb{R}^n$ in the weak sense.
  - ⋆ As u_ε and f_ε are smooth, this equation also holds in the classical sense. (Check this!)
- Now, by the previous lemma, we have

$$\|\nabla^2 u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \|\Delta u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \|f_\varepsilon\|_{L^2(\mathbb{R}^n)}.$$

- Young's convolution inequality gives $\|f_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}\|\varrho_\varepsilon\|_{L^1(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$, and so

$$\|\nabla^2 u_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(B_2)}.$$

- Therefore, along a subsequence, $(\nabla^2 u_\varepsilon)$ converges weakly to some $A \in L^2(\mathbb{R}^n; \mathbb{R}^{n\times n})$ with $\|A\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(B_2)}$.

The case of $-\Delta$

Proof of the lemma

- Putting things together we have $u_\varepsilon \to u$ in $H^1(\mathbb{R}^n)$, $\nabla^2 u_\varepsilon \rightharpoonup A$ in $L^2(\mathbb{R}^n)$ and $\|A\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}$.
- Claim: A is the weak second derivatives of u.
  Indeed, this follows by passing $\varepsilon \to 0$ in the identity

$$\int_{\mathbb{R}^n} u_\varepsilon \partial_i \partial_j v = \int_{\mathbb{R}^n} \partial_i \partial_j u_\varepsilon v \text{ for all } v \in C_c^\infty(\mathbb{R}^n).$$

- We have thus shown that $u \in H^2(\mathbb{R}^n)$ and and $\|\nabla^2 u\|_{L^2(\mathbb{R}^n)} = \|A\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(B_2)}$.

The case of $-\Delta$

Proof of interior H^2 regularity for $-\Delta$.

- Step 1: Reduction to regularity estimates for solutions which vanish near ∂B_2.
  - ⋆ We do a truncation: Fix a cut-off function $\zeta \in C_c^\infty(B_2)$ such that $\zeta \equiv 1$ in B_1 and consider $w := \zeta u$.
  - ⋆ Formally, if we have enough regularity, we have

$$\begin{aligned} -\Delta w = -\Delta(\zeta u) &= -\zeta \Delta u - 2\nabla\zeta \cdot \nabla u - u\Delta\zeta \\ &= (\zeta f - \nabla\zeta \cdot \nabla u) - \partial_i(u \partial_i \zeta). \end{aligned}$$

  - ⋆ We claim that the above formula for $-\Delta w$ is valid in the weak sense, i.e.

$$\int_{B_2} \nabla w \cdot \nabla v \, dx = \int_{B_2} \Big[(\zeta f - \nabla\zeta \cdot \nabla u) v + u \nabla\zeta \cdot \nabla v\Big] \, dx \text{ for all } v \in H_0^1(B_2).$$

Proof of interior H^2 regularity for $-\Delta$.

- Step 1: Reduction to regularity estimates for solutions which vanish near ∂B_2.
  - ... Claim:

$$\int_{B_2} \nabla w \cdot \nabla v \, dx = \int_{B_2} \Big[(\zeta f - \nabla \zeta \cdot \nabla u) v + u \nabla \zeta \cdot \nabla v\Big] \, dx \text{ for all } v \in H_0^1(B_2).$$

  - Moving the last term on the right hand side to the left hand side and using $w = \zeta u$, we need to check that

$$\int_{B_2} \zeta \nabla u \cdot \nabla v = \int_{B_2} (\zeta f - \nabla \zeta \cdot \nabla u) v \text{ for all } v \in H_0^1(B_2).$$

  - Moving the last term on the right hand side to the left hand side once again, we then need to check

$$\int_{B_2} \nabla u \cdot \nabla(\zeta v) = \int_{B_2} f(\zeta v) \text{ for all } v \in H_0^1(B_2),$$

But this is true as $\zeta v \in H_0^1(B_2)$ and $-\Delta u = f$ in B_2 weakly.

The case of $-\Delta$

Proof of interior H^2 regularity for $-\Delta$.

- Step 1: Reduction to regularity estimates for solutions which vanish near ∂B_2.
  - ⋆ We have thus proved the claim that

$$-\Delta w = (\zeta f - \nabla\zeta \cdot \nabla u) - \partial_i(u\partial_i\zeta) \text{ in } B_2$$

    in the weak sense.
  - ⋆ Now if the interior H^2 estimate has been established for functions which vanish near the boundary, then by applying such estimate to w, we get $w \in H^2(B_1)$ and

$$\begin{aligned}\|w\|_{H^2(B_1)} &\leq C(\|(\zeta f - \nabla\zeta \cdot \nabla u) - \partial_i(u\partial_i\zeta)\|_{L^2(B_2)} + \|w\|_{H^1(B_2)}) \\ &\leq C(\|f\|_{L^2(B_2)} + \|u\|_{H^1(B_2)}).\end{aligned}$$

    As $u = w$ in B_1, we thus have the desired estimate for u in B_1.

The case of $-\Delta$

Proof of interior H^2 regularity for $-\Delta$.

- Step 2: Reduction to estimates on the whole space.
  - ⋆ Suppose that $u \in H^1_0(B_2)$ vanishes near ∂B_2 and satisfies $-\Delta u = f$ in B_2 in the weak sense for some $f \in L^2(B_2)$. We would like to bound $\|u\|_{H^2(B_1)}$.
  - ⋆ We extend u and f by zero outside of B_2 so that $u \in H^1_0(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$.
  - ⋆ Claim: $-\Delta u = f$ on $\mathbb{R}^n$, i.e.

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx = \int_{\mathbb{R}^n} f \, v \, dx \text{ for all } v \in C_c^\infty(\mathbb{R}^n).$$

  - ⋆ If v has support in B_2, this is true because $-\Delta u = f$ in B_2. For general v, we will truncate v exploiting the fact that u vanishes near ∂B_2, say $u \equiv 0$ in $B_2 \setminus B_R$ for some $R < 2$. Note that we also have $f = 0$ in $B_2 \setminus B_R$ (check this!).

The case of $-\Delta$

Proof of interior H^2 regularity for $-\Delta$.

- Step 2: Reduction to estimate on the whole space.
  - ⋆ Take a cut-off function ζ such that $\zeta \equiv 1$ in B_R and $\zeta \equiv 0$ in $\mathbb{R}^n \setminus B_2$. Then, for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx = \int_{B_2} \nabla u \cdot \nabla(\zeta v) \, dx$$
$$= \int_{B_2} f \, \zeta \, v \, dx = \int_{\mathbb{R}^n} f \, v \, dx.$$

    Hence $-\Delta u = f$ in $\mathbb{R}^n$.
  - ⋆ Now, by the lemma, we have $\nabla^2 u \in L^2(\mathbb{R}^n)$ and

$$\|\nabla^2 u\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}.$$

    which gives the conclusion.

A priori H^2 estimates in the general case

- We now turn to the case where a is variable. To better convey central ideas, we will focus in the rest of this course to a priori estimates: We assume that the solution has the right regularity and will be concerned with establishing quantitative estimates.
- More precisely, we suppose that u belongs to $\underline{H^2(\mathbb{R}^n)}$ and is a weak solution to $Lu = f$ in $\mathbb{R}^n$, and would like to bound $\|u\|_{H^2(\mathbb{R}^n)}$ in terms of the bounds for the coefficients of L, $\|f\|_{L^2(\mathbb{R}^n)}$ and $\|u\|_{H^1(\mathbb{R}^n)}$.
- For simplicity, we will assume that $b \equiv 0$ and $c \equiv 0$. You should check that the methods we use work in the general case.

Method of freezing coefficients

Theorem

Suppose $a \in C^1(\mathbb{R}^n)$, $\nabla a \in L^\infty(\mathbb{R}^n)$ *and* $L = -\partial_i(a_{ij}\partial_j)$.
There exist $0 < \delta_0 \ll 1$ *and* $C > 0$ *such that if* $\|a_{ij} - \delta_{ij}\|_{L^\infty(\mathbb{R}^n)} \leq \delta_0$ *and if* $u \in H^2(\mathbb{R}^n)$ *and satisfies* $Lu = f$ *in* $\mathbb{R}^n$ *in the weak sense, then*

$$\|u\|_{H^2(\mathbb{R}^n)} \leq C(\|f\|_{L^2(\mathbb{R}^n)} + \|u\|_{H^1(\mathbb{R}^n)}).$$

Proof

- Claim: u satisfies

$$-\Delta u = f + (a_{ij} - \delta_{ij})\partial_i\partial_j u + \partial_i a_{ij}\partial_j u =: \tilde{f},$$

that is, for all $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx = \int_{\mathbb{R}^n} \Big[f + (a_{ij} - \delta_{ij})\partial_i\partial_j u + \partial_i a_{ij}\partial_j u\Big] v \, dx.$$

Proof

- Claim: for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx = \int_{\mathbb{R}^n} \Big[f + (a_{ij} - \delta_{ij}) \partial_i \partial_j u + \partial_i a_{ij} \partial_j u \Big] v \, dx.$$

  - ⋆ We note that $(a_{ij} - \delta_{ij}) v \in C_c^1(\mathbb{R}^n)$. Hence, by definition of weak derivatives,

$$\begin{aligned}
\int_{\mathbb{R}^n} (a_{ij} - \delta_{ij}) \partial_i \partial_j u v \, dx &= - \int_{\mathbb{R}^n} \partial_j u \partial_i [(a_{ij} - \delta_{ij}) v] \, dx \\
&= - \int_{\mathbb{R}^n} \partial_j u [(a_{ij} - \delta_{ij}) \partial_i v + \partial_i a_{ij} v] \, dx \\
&= \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx - \int_{\mathbb{R}^n} a_{ij} \partial_j u \partial_i v \, dx \\
&\qquad - \int_{\mathbb{R}^n} \partial_i a_{ij} v \, dx.
\end{aligned}$$

Method of freezing coefficients

Proof

- Claim: for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v\,dx = \int_{\mathbb{R}^n} \Big[f + (a_{ij} - \delta_{ij})\partial_i\partial_j u + \partial_i a_{ij}\partial_j u\Big] v\,dx.$$

  - ⋆ $$\int_{\mathbb{R}^n} (a_{ij} - \delta_{ij})\partial_i\partial_j uv\,dx = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v\,dx - \int_{\mathbb{R}^n} a_{ij}\partial_j u\partial_i v\,dx - \int_{\mathbb{R}^n} \partial_i a_{ij} v\,dx.$$
  - ⋆ As $Lu = f$, we have

$$\int_{\mathbb{R}^n} a_{ij}\partial_j u\partial_i v\,dx = \int_{\mathbb{R}^n} f\,v\,dx.$$

  - ⋆ Putting the two identities together, we obtain the claim.

Method of freezing coefficients

Proof

- We have proved the claim that $-\Delta u = \tilde{f} = f + (a_{ij} - \delta_{ij})\partial_i\partial_j u + \partial_i a_{ij}\partial_j u$.
- By the lemma on the H^2 regularity for $-\Delta$, we have a constant C such that

$$\begin{aligned}\|\nabla^2 u\|_{L^2} &\leq C\|\tilde{f}\|_{L^2}\\ &\leq C\Big[\|f\|_{L^2} + \|a_{ij} - \delta_{ij}\|_{L^\infty}\|\nabla^2 u\|_{L^2(\Omega)}\\ &\qquad + \|\partial_i a_{ij}\|_{L^\infty}\|\nabla u\|_{L^2}\Big].\end{aligned}$$

- It is readily seen that if $C\|a_{ij} - \delta_{ij}\|_{L^\infty} < 1$, then the second term on the right hand side can be absorbed back to the left hand side, giving the conclusion:

$$\|\nabla^2 u\|_{L^2} \leq C'\Big[\|f\|_{L^2} + \|\nabla u\|_{L^2}\Big].$$

Method of differentiating the equation

Theorem

Suppose $a \in C^1(\mathbb{R}^n)$, $\nabla a \in L^\infty(\mathbb{R}^n)$ *and* $L = -\partial_i(a_{ij}\partial_j)$.
There exists $C > 0$ *such that if* $u \in H^2(\mathbb{R}^n)$ *and satisfies* $Lu = f$ *in* $\mathbb{R}^n$ *in the weak sense, then*

$$\|u\|_{H^2(\mathbb{R}^n)} \leq C(\|f\|_{L^2(\mathbb{R}^n)} + \|u\|_{H^1(\mathbb{R}^n)}).$$

Proof

- Let $w = \partial_k u \in H^1(\mathbb{R}^n)$. We would like to bound $\|w\|_{H^1}$.
- Claim: w satisfies

$$Lw = \partial_i h_i \text{ where } h_i = \partial_k a_{ij}\partial_j u + f\,\delta_{ik},$$

that is, for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v\,dx = -\int_{\mathbb{R}^n} [\partial_k a_{ij}\partial_j u + f\,\delta_{ik}]\partial_i v\,dx.$$

Method of differentiating the equation

Proof

- Claim: for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v\, dx = -\int_{\mathbb{R}^n} [\partial_k a_{ij}\partial_j u + f\, \delta_{ik}]\partial_i v\, dx.$$

  ⋆ Note that $a_{ij}\partial_i v \in C_c^1(\mathbb{R}^n)$. Hence, by definition of weak derivatives,

$$\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v\, dx = \int_{\mathbb{R}^n} \partial_k \partial_j u\,(a_{ij}\partial_i v)\, dx = -\int_{\mathbb{R}^n} \partial_j u\, \partial_k(a_{ij}\partial_i v)\, dx$$
$$= -\int_{\mathbb{R}^n} a_{ij}\partial_j u\, \partial_k \partial_i v\, dx - \int_{\mathbb{R}^n} \partial_j u\, \partial_k a_{ij}\partial_i v\, dx$$

Method of differentiating the equation

Proof

- Claim: for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v\, dx = -\int_{\mathbb{R}^n} [\partial_k a_{ij}\partial_j u + f\,\delta_{ik}]\partial_i v\, dx.$$

  ⋆ $\displaystyle\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v\, dx = -\int_{\mathbb{R}^n} a_{ij}\partial_j u\,\partial_k\partial_i v\, dx - \int_{\mathbb{R}^n} \partial_j u\,\partial_k a_{ij}\partial_i v\, dx.$

  ⋆ On the other hand, using $\partial_k v$ as a test function for $Lu = f$, we have
$$\int_{\mathbb{R}^n} a_{ij}\partial_j u\,\partial_i\partial_k v\, dx = \int_{\mathbb{R}^n} f\partial_k v\, dx.$$

  ⋆ Putting the two identities together we get the claim.

Method of differentiating the equation

Proof

- We have thus shown that $Lw = \partial_i h_i$ with $h_i = \partial_k a_{ij} \partial_j u + f\,\delta_{ik}$.
- Using w as a test function for this equation, we get

$$\int_{\mathbb{R}^n} a_{ij} \partial_j w \partial_i w \, dx = -\int_{\mathbb{R}^n} h_i \partial_i w \, dx.$$

- Using ellipticity on the left side and Cauchy-Schwarz' inequality on the right side we get

$$\lambda \|\nabla w\|_{L^2}^2 \leq \|h\|_{L^2} \|\nabla w\|_{L^2} \leq \frac{\lambda}{2} \|\nabla w\|_{L^2}^2 + \frac{1}{2\lambda} \|h\|_{L^2}^2.$$

- We thus have

$$\|\nabla w\|_{L^2} \leq C \|h\|_{L^2} \leq C \Big[\|f\|_{L^2} + \|\nabla u\|_{L^2} \Big].$$

Recalling that $w = \partial_k u$, we're done.