C4.3 Functional Analytic Methods for PDEs
Lecture 15

Luc Nguyen
luc.nguyen@maths

University of Oxford

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In the last lecture

- H^2 regularity of weak solutions to linear elliptic equations.

This lecture

- Continuity of weak solutions to linear elliptic equations.

Example

- Recall the example of the equation $-(au')' = f$ in $(-1, 1)$ with $a = \chi_{(-1,0)} + 2\chi_{(0,1)}$.
- If $f \in L^q$, then $au' \in W^{1,q}$ and so u' is presumably discontinuous.
- Nevertheless as u' exists by assumption, u is continuous.
- In higher dimension, the existence of ∇u (in L^2) doesn't ensure continuity of u. Nevertheless, a major result due to De Giorgi, Moser and Nash around late 50s asserts that u is indeed continuous!

De Giorgi-Moser-Nash's theorem

Theorem (De Giorgi-Moser-Nash's theorem)

Suppose that $a, b, c \in L^\infty(\Omega)$*, a is uniformly elliptic, and* $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$*. If* $u \in H^1(\Omega)$ *satisfies* $Lu = f$ *in* Ω *in the weak sense for some* $f \in L^q(\Omega)$ *with* $q > \frac{n}{2}$*, then u is locally Hölder continuous , and for any open* ω *such that* $\bar{\omega} \subset \Omega$ *we have*

$$\|u\|_{C^{0,\alpha}(\omega)} \leq C(\|f\|_{L^q(\Omega)} + \|u\|_{H^1(\Omega)})$$

where the constant C *depends only on* $n, \Omega, \omega, a, b, c$*, and the Hölder exponent* α *depends only on* n, Ω, ω, a*.*

A digression

We make some observations:

- In De Giorgi-Moser-Nash's theorem, no continuity is assumed on the coefficients a_{ij}.
- If a_{ij} is continuous, one can imagine using the method of freezing coefficients to reduce to the case a_{ij} is constant. Hence the model equation is $-\Delta u = f$.
- In $1d$, we have $-u'' = f$. If $f \in L^q$, we then have that $u \in W^{2,q}_{loc}$.
- It turns out that, in any dimension, if $-\Delta u = f$ and $f \in L^q$, then $u \in W^{2,q}_{loc}$.
  In particular, when $n/2 < q < n$, by the embedding $W^{2,q}_{loc} \hookrightarrow W^{1,\frac{qn}{n-q}}_{loc} \hookrightarrow C^{0,2-\frac{n}{q}}_{loc}$, we have u is Hölder continuous.

Global a priori L^∞ estimate

To illustrate the method, we will assume for simplicity that $b \equiv 0$ and $c \equiv 0$. We will focus on a priori L^∞ estimates, i.e. we assume that the solution $u \in L^\infty$ and try to establish estimates for $\|u\|_{L^\infty}$.

- We assume in addition for now a boundary condition: $u = 0$ on ∂B_1.

Theorem (Global a priori L^∞ estimates)

Suppose that $a \in L^\infty(B_1)$, a is uniformly elliptic, $b \equiv 0$, $c \equiv 0$ and $L = -\partial_i(a_{ij}\partial_j)$. If $u \in H_0^1(B_1) \cap L^\infty(B_1)$ satisfies $Lu = f$ in B_1 in the weak sense and $f \in L^q(B_1)$ with $q > n/2$, then

$$\|u\|_{L^\infty(B_1)} \leq C(\|f\|_{L^q(B_1)} + \|u\|_{L^2(B_1)})$$

where the constant C depends only on n, q, a.

Truncations and powers of H^1 functions

Lemma

Suppose that $u \in H_0^1(B_1) \cap L^\infty(B_1)$. Then, for $p \geq 1$ and $k \geq 0$, one has $(u_+ + k)^p - k^p \in H_0^1(B_1)$.

Proof

- As $u \in L^\infty(B_1)$, we can suppose $|u| \leq M$ a.e. in B_1.
- By Sheet 3, $u_+ \in H^1(B_1)$.
- Select a function $g \in C^1(\mathbb{R})$ such that $g(t) = (t_+ + k)^p - k^p$ for $t \leq M$, and $g(t) = (M + k + 1)^p - k^p$ for $t \geq M + 1$. Note that $(u_+ + k)^p - k^p = g(u)$.
- Then $|g(t)| + |g'(t)| \leq C$ on $\mathbb{R}$.
- By the chain rule (Sheet 2), $g(u)$ has weak derivatives $\nabla g(u) = g'(u) \nabla u \in L^2(B_1)$. Hence $g(u) \in H^1(B_1)$.

Truncations and powers of H^1 functions

Proof

- $g(u) \in H^1(B_1)$.
- We next show that $g(u) \in H^1_0(B_1)$.
  Approximate u by $(u_m) \in C^\infty_c(B_1)$. The argument above shows that $g(u_m) \in H^1(B_1)$.
  As $g(u_m)$ is continuous, we have that the its trace on ∂B_1 is zero, hence $g(u_m) \in H^1_0(B_1)$.
- We have, by Lebesgue's dominated convergence theorem

$$\int_{B_1} |g(u_m) - g(u)|^2 \, dx \to 0.$$

  So $g(u_m) \to g(u)$ in L^2.

Truncations and powers of H^1 functions

Proof

- Next, we have

$$\begin{aligned}\int_B |\nabla g(u_m) - \nabla g(u)|^2\,dx &= \int_B |g'(u_m)\nabla u_m - g'(u)\nabla u|^2\,dx\\ &\leq \int_B |g'(u_m) - g'(u)|^2 |\nabla u|^2\,dx\\ &\qquad + \int_B |g'(u_m)|^2 |\nabla u_m - \nabla u|^2\,dx \to 0,\end{aligned}$$

  where we use Lebesgue's dominated convergence theorem to treat the first integral and the convergence of ∇u_m to ∇u in L^2 to treat the second integral.
  Hence $\nabla g(u_m) \to \nabla g(u)$ in L^2.
- We have thus shown that $g(u_m) \in H^1_0(B)$ and $g(u_m) \to g(u)$ in $H^1(B)$. The conclusion follows.

Global a priori L^∞ estimates

We now prove the statement that if $u \in H_0^1(B_1) \cap L^\infty(B_1)$ is such that $Lu = f$ in B_1 with $f \in L^q(B_1)$ for some $q > n/2$, then

$$\|u\|_{L^\infty(B_1)} \leq C(\|f\|_{L^q(B_1)} + \|u\|_{L^2(B_1)}).$$

- We use Moser iteration method. We write $B = B_1$ and fix some $k > 0$, $p \geq 1$.
- Let $w = u_+ + k$ and we use $v = w^p - k^p$ as test function. This is possible because we just proved that $v \in H_0^1(B_1)$.
  We have

$$\begin{aligned} \int_B f\, v\, dx &= \int_B a_{ij}\partial_j u \partial_i v\, dx \\ &= \int_B p w^{p-1} a_{ij}\partial_j u \partial_i u_+\, dx \\ &\overset{ellipticity}{\geq} \lambda p \int_B w^{p-1} |\nabla u_+|^2\, dx. \end{aligned}$$

Global a priori L^∞ estimate

Proof

- We thus have
$$\int_B |\nabla w^{\frac{p+1}{2}}|^2\,dx \le Cp\int_B |f|\,|v|\,dx \le Cp\int_B |f|w^p\,dx.$$
- By Friedrichs' inequality, this gives
$$\|w^{\frac{p+1}{2}} - k^{\frac{p+1}{2}}\|_{H^1}^2 \le Cp\int_B |f|w^p\,dx.$$
- By Gagliardo-Nirenberg-Sobolev's inequality, this implies that
$$\|w^{\frac{p+1}{2}} - k^{\frac{p+1}{2}}\|_{L^{\frac{2n}{n-2}}}^2 \le Cp\int_B |f|\,w^p\,dx.$$
- We thus have
$$\|w^{\frac{p+1}{2}}\|_{L^{\frac{2n}{n-2}}}^2 \le Cp\int_B \Big(\frac{|f|}{k} + 1\Big)\,w^{p+1}\,dx.$$

Global a priori L^∞ estimate

Proof

- $\|w^{\frac{p+1}{2}}\|^2_{L^{\frac{2n}{n-2}}} \leq Cp \int_B (\frac{|f|}{k} + 1)\, w^{p+1}\, dx.$
- Using Hölder's inequality, we then arrive at

$$\|w^{p+1}\|_{L^{\frac{n}{n-2}}} \leq Cp(\|\frac{|f|}{k}\|_{L^q} + 1)\|w^{p+1}\|_{L^{q'}}.$$

- We now choose k to be any number larger than $\|f\|_{L^q}$ and obtain from the above that

$$\|w\|^{p+1}_{L^{\frac{n(p+1)}{n-2}}} \leq Cp\|w\|^{p+1}_{L^{q'(p+1)}}.$$

Recalling that $q > n/2$, we have $q' < \frac{n}{n-2}$. Thus the above inequality is self-improving: If w has a bound in $L^{q'(p+1)}$, then it has a bound in $L^{\frac{n(p+1)}{n-2}}$.

Global a priori L^∞ estimate

Proof

- $\|w\|^{p+1}_{L^{\frac{n(p+1)}{n-2}}} \leq C(p+1)\|w\|^{p+1}_{L^{q'(p+1)}}$.
- Now let $\chi = \frac{n}{(n-2)q'} > 1$ and $t_m = \gamma\chi^m$ for some $\gamma > 2q'$, then the above gives

$$\begin{aligned}\|w\|_{L^{t_{m+1}}} &\leq (Ct_m)^{\frac{q'}{t_m}}\|w\|_{L^{t_m}}\\ &= (C\gamma)^{q'\gamma^{-1}\chi^{-m}}\chi^{q'\gamma^{-1}m\chi^{-m}}\|w\|_{L^{t_m}}.\end{aligned}$$

  Hence by induction,

$$\|w\|_{L^{t_{m+1}}} \leq (C\gamma)^{q'\gamma^{-1}\sum_m \chi^{-m}}\chi^{q'\gamma^{-1}\sum_m m\chi^{-m}}\|w\|_{L^\gamma} \leq C\|w\|_{L^\gamma}.$$

- Sending $m \to \infty$, we obtain

$$\|w\|_{L^\infty} \leq C\|w\|_{L^\gamma} \text{ provided } \gamma > 2q'.$$

Global a priori L^∞ estimate

Proof

- $\|w\|_{L^\infty} \leq C\|w\|_{L^\gamma}$ when $\gamma > 2q'$.
- We now reduce from L^γ to L^2:

$$\|w\|_{L^\infty} \leq C\Big\{\int_B |w|^\gamma\,dx\Big\}^{1/\gamma} \leq C\|w\|_{L^\infty}^{1-\frac{2}{\gamma}}\Big\{\int_B |w|^2\,dx\Big\}^{1/\gamma}.$$

  This gives

$$\|w\|_{L^\infty} \leq C\|w\|_{L^2}.$$

- Recalling that $w = u_+ + k$ and k can be any positive constant larger than $\|f\|_{L^q}$, we have thus shown that

$$\|u_+\|_{L^\infty} \leq C(\|u\|_{L^2} + \|f\|_{L^q})$$

- Applying the same argument to u_-, we get the corresponding bound for u_- and conclude the proof.

Global a priori L^∞ estimate

Remark

When L is injective, the term $\|u\|_{L^2(B_1)}$ on the right hand side can be dropped yielding the estimate:

$$\|u\|_{L^\infty(B_1)} \leq C\|f\|_{L^q(B_1)}.$$

We knew that

$$\|u\|_{L^\infty} \leq C(\|f\|_{L^q} + \|u\|_{L^2}).$$

Therefore, it suffices to show that

$$\|u\|_{L^2} \leq C\|f\|_{L^q} \text{ for all } u \in H^1_0(B_1), f \in L^q(B_1) \text{ with } Lu = f.$$

Energy estimate with L^q right hand side

Theorem

Suppose that $a, b, c \in L^\infty(B_1)$, a is uniformly elliptic, and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. Suppose that the only solution in $H_0^1(B_1)$ to $Lu = 0$ is the trivial solution. Then, for every $u \in H_0^1(B_1)$ and $f \in L^q(B_1)$ with $q \geq \frac{2n}{n+2}$ satisfying $Lu = f$ in B_1, there holds

$$\|u\|_{H^1(B_1)} \leq C\|f\|_{L^q(B_1)}$$

where the constant C depends only on n, q, a, b, c.

Proof

- When $q = 2$, the result is a consequence of the Fredholm alternative and the inverse mapping theorem.

Energy estimate with L^q right hand side

Proof

- Let us consider first the case that $b \equiv 0$ and $c \equiv 0$.
  - ⋆ In this case, by using u as a test function, we have

$$\lambda \|\nabla u\|_{L^2}^2 \leq \int_{B_1} a_{ij} \partial_j u \partial_i u \, dx = \int_B f u \, dx \leq \|f\|_{L^q} \|u\|_{L^{q'}}.$$

  - ⋆ By Friedrichs' inequality, we have $\|u\|_{H^1} \leq C \|\nabla u\|_{L^2}$. As $q \geq \frac{2n}{n+2}$, $q' \leq \frac{2n}{n-2}$. Hence, by Gagliardo-Nirenberg-Sobolev's inequality, $\|u\|_{L^{q'}} \leq C \|u\|_{H^1}$.
  - ⋆ Therefore

$$\|u\|_{H^1}^2 \leq C \|\nabla u\|_{L^2}^2 \leq C \|f\|_{L^q} \|u\|_{L^{q'}} \leq C \|f\|_{L^q} \|u\|_{H^1},$$

from which we get $\|u\|_{H^1} \leq C \|f\|_{L^q}$, as desired.

Energy estimate with L^q right hand side

Proof

- Let us now consider the general case. By using u as a test function, we have

$$B(u,u) = \int_{B_1} fu\,dx \leq \|f\|_{L^q}\|u\|_{L^{q'}},$$

  where B is the bilinear form associated with L.

- The right hand side is treated as before and is bounded from above by $C\|f\|_{L^q}\|u\|_{H^1}$. For the left hand side, we use Friedrichs' inequality together with energy estimates:

$$B(u,u) + C\|u\|_{L^2}^2 \geq \frac{\lambda}{2}\|\nabla u\|_{L^2}^2 \geq \frac{1}{C}\|u\|_{H^1}^2.$$

  We thus have

$$\|u\|_{H^1}^2 \leq C\|f\|_{L^q}\|u\|_{H^1} + C\|u\|_{L^2}^2.$$

Energy estimate with L^q right hand side

Proof

- $\|u\|_{H^1}^2 \leq C\|f\|_{L^q}\|u\|_{H^1} + C\|u\|_{L^2}^2$.
- By Cauchy-Schwarz' inequality, we then have

$$\|u\|_{H^1}^2 \leq \frac{1}{2}\|u\|_{H^1}^2 + C\|f\|_{L^q}^2 + C\|u\|_{L^2}^2,$$

  and so

$$\|u\|_{H^1}^2 \leq C\|f\|_{L^q}^2 + C\|u\|_{L^2}^2.$$

- In other words,

$$\|u\|_{H^1} \leq C\|f\|_{L^q} + C\|u\|_{L^2}. \tag{*}$$

- To conclude, we show that

$$\|u\|_{L^2} \leq C\|f\|_{L^q}. \tag{**}$$

  More precisely, we show that "(*) + injectivity of L $\Rightarrow$ (**)".

Energy estimate with L^q right hand side

Proof

- Suppose by contradiction that there exists sequence $u_m \in H_0^1(B_1)$, $f_m \in L^q(B_1)$ such that $Lu_m = f_m$ but

$$\|u_m\|_{L^2} > m\|f_m\|_{L^q}.$$

  Replacing u_m by $\frac{1}{\|u_m\|_{L^2}} u_m$ if necessary, we can assume that $\|u_m\|_{L^2} = 1$.
- Then $\|u_m\|_{L^2} = 1$, $\|f_m\|_{L^q} < \frac{1}{m}$ and by (*), $\|u_m\|_{H^1} \leq C$. By the reflexivity of H^1 and Rellich-Kondrachov's theorem, we may assume that $u_m \rightharpoonup u$ in H^1 and $u_m \to u$ in L^2. Note that $\|u\|_{L^2} = 1$.
- To conclude, we show that $Lu = 0$, which implies $u = 0$ by hypothesis, and amounts to a contradiction with $\|u\|_{L^2} = 1$.

Energy estimate with L^q right hand side

Proof

- We start with $Lu_m = f_m$ which means

$$\int_{B_1} \Big[a_{ij}\partial_j u_m \partial_i v + b_i \partial_i u_m v + c u_m v \Big]\, dx = \int_{B_1} f_m v \, dx \text{ for all } v \in H^1_0(B_1).$$

  We then send $m \to \infty$ using that $\nabla u_m \rightharpoonup \nabla u$ in L^2, $u_m \to u$ in L^2 and $f_m \to 0$ in L^q to obtain

$$\int_{B_1} \Big[a_{ij}\partial_j u \partial_i v + b_i \partial_i u v + c u v \Big]\, dx = 0 \text{ for all } v \in H^1_0(B_1),$$

  i.e. $Lu = 0$, as desired.

- As $u_m \in H^1_0(B_1)$, we have $u \in H^1_0(B_1)$ and so $u = 0$ by hypothesis. This contradicts the identity $\|u\|_{L^2} = 1$, and finishes the proof.

A case study

Let us now consider an example in $1d$:

$$\begin{cases} -(au')' = f \text{ in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \qquad \text{where } a = \chi_{(-1,0)} + k\chi_{(0,1)}.$$

As $k \to 0$, the ellipticity deteriorates. As $k \to \infty$, the boundedness of k deteriorates.
We have proved 2 estimates:

$$\|u\|_{L^\infty(-1,1)} \leq C_1(k)\|f\|_{L^\infty(-1,1)}, \tag{1}$$

$$\|u\|_{L^\infty(-1,1)} \leq C_2(k)\big(\|f\|_{L^\infty(-1,1)} + \|u\|_{L^2(-1,1)}\big). \tag{2}$$

We would now like to have a rough appreciation whether (or how) these constants depend on k, as $k \to 0$ or ∞.

A case study

$$\begin{cases} -(au')' = f \text{ in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \qquad \text{where } a = \chi_{(-1,0)} + k\chi_{(0,1)}.$$

- We empirically take $f = 1$, so that $\|f\|_{L^\infty} = 1$.
- We know that the problem has uniqueness (why?), so it suffices to find a solution.
- The equation gives $-u'' = 1$ in $(-1,0)$ and $-u'' = 1/k$ in $(0,1)$. So u takes the form

$$u(x) = \begin{cases} -\frac{1}{2}(x+1)^2 + \alpha(x+1) & \text{for } x \in (-1,0), \\ -\frac{1}{2k}(x-1)^2 + \beta(x-1) & \text{for } x \in (0,1). \end{cases}$$

A case study

$$\begin{cases} -(au')' = 1 \text{ in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \qquad \text{where } a = \chi_{(-1,0)} + k\chi_{(0,1)}.$$

- As $u \in H^1(-1,1)$, u is continuous. So

$$-\frac{1}{2} + \alpha = -\frac{1}{2k} - \beta.$$

- As au' is weakly differentiable, it is continuous and so

$$-1 + \alpha = 1 + k\beta.$$

- So we find $\alpha = \frac{k+3}{2(k+1)}$ and $\beta = -\frac{3k+1}{2k(k+1)}$.

A case study

$$\begin{cases} -(au')' = 1 \text{ in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \qquad \text{where } a = \chi_{(-1,0)} + k\chi_{(0,1)}.$$

- So we have

$$u(x) = \begin{cases} -\frac{1}{2}(x+1)^2 + \frac{k+3}{2(k+1)}(x+1) & \text{for } x \in (-1,0), \\ -\frac{1}{2k}(x-1)^2 - \frac{3k+1}{2k(k+1)}(x-1) & \text{for } x \in (0,1). \end{cases}$$

- We find $\|u\|_{L^\infty} \sim \frac{1}{k}$ as $k \to 0$, and $\|u\|_{L^\infty} \sim 1$ as $k \to \infty$. Therefore

$$C_1(k) \sim \frac{1}{k} \text{ as } k \to 0, \text{ and } C_1(k) \sim 1 \text{ as } k \to \infty.$$

- Similarly $\|u\|_{L^2} \sim \frac{1}{k}$ as $k \to 0$, and $\|u\|_{L^2} \sim 1$ as $k \to \infty$. Therefore

$$C_2(k) \sim 1 \text{ as } k \to 0, \infty.$$

More examples...

Some other motivating examples you may want to consider:
$a = \chi_{(-1,1)\setminus A} + k\chi_A$ where

- A is an interval of length ε.
- A consists of two or more disjoint intervals of distance ε apart.

Studies of this kind in higher dimensions are active area of research, due to their practical importance.