C4.3 Functional Analytic Methods for PDEs - Sheet 3 of 4

Read Chapter 3 and prove the few statements whose proofs were left out as exercises. (Not to be handed in.)

Do:

Q1. Let Ω be a bounded domain in $\mathbb{R}^n$ and assume that $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$. By considering the sequence $\sqrt{m^{-2} + u^2}$, or otherwise, show that $|u| \in W^{1,p}(\Omega)$ and has weak derivative

$$(D|u|)(x) = \begin{cases} Du(x) & \text{if } u(x) > 0, \\ -Du(x) & \text{if } u(x) < 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

Q2. Suppose that Ω be a bounded domain in $\mathbb{R}^n$ and $u \in W^{1,1}(\Omega)$.

(i) Let U be an open subset of $\mathbb{R}$ and define $G_U = \int_0^t \chi_U(s)\,ds$. Show that, for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \chi_U(u(x)) Du(x) \varphi(x)\,dx = -\int_\Omega G_U(u(x)) D\varphi(x)\,dx.$$

[*Start by computing the weak derivative of* $(u-a)^+$, *for some constant* a, *using Q1, then successively consider the cases* $U = (a, \infty)$, $U = (a, b)$, *and finally any open* U.]

(ii) Let $Z \subset \mathbb{R}$ be a measurable set of zero measure and let $B = u^{-1}(Z)$. Show that $Du = 0$ a.e. in B. [*Select a sequence of open sets* $U_1 \supset U_2 \supset \ldots \supset Z$ *such that* $|U_m| \to 0$. *Apply (i) to each* U_m *and send* $m \to \infty$.]

Q3. Let Ω be a bounded domain in $\mathbb{R}^n$ and $1 \leq p < n$. Suppose that for some $1 \leq q \leq \infty$ the following Sobolev-type inequality holds

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \text{ for all } u \in W^{1,p}(\Omega),$$

where C is independent of u. Show that $q \leq p^*$.

Q4. (Sobolev spaces in one dimension) Let $1 \le p < \infty$.

(i) Let $v \in L^p(0,1)$. Show that the function $w : [0,1] \to \mathbb{R}$ defined by $w(x) = \int_0^x v(t)\,dt$ is continuous and has weak derivative $w' = v$.

(ii) Show that a function $u \in L^p(0,1)$ belongs to $W^{1,p}(0,1)$ if and only if there exist a function $v \in L^p(0,1)$ and a constant c such that $u(x) = c + \int_0^x v(t)\,dt$ for a.e. $x \in (0,1)$.

(iii) Deduce that the embedding $W^{1,p}(0,1) \hookrightarrow C([0,1])$ is continuous.

(iv) For $p > 1$, prove the inequality

$$[u]_{C^{0,1-\frac{1}{p}}(0,1)} \le \|u'\|_{L^p(0,1)} \text{ for all } u \in W^{1,p}(0,1)$$

and deduce that the embedding $W^{1,p}(0,1) \hookrightarrow C([0,1])$ is compact.

Q5. (Poincaré-Sobolev's inequality) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and suppose $1 \le p < \infty$. Let $1 \le q \le p^*$ when $p < n$, $1 \le q < \infty$ when $p = n$, and $1 \le q \le \infty$ when $p > n$. Show that there exists a constant $C = C_{n,p,q,\Omega}$ such that

$$\|u - \bar{u}_\Omega\|_{L^q(\Omega)} \le C\|Du\|_{L^p(\Omega)} \text{ for all } u \in W^{1,p}(\Omega).$$

Here $\bar{u}_\Omega = \frac{1}{|\Omega|}\int_\Omega u$.

Q6. (i) (Gagliardo-Nirenberg's inequality in two dimensions) Prove that

$$\|u\|^2_{L^2(\mathbb{R}^2)} \le \|\partial_1 u\|_{L^1(\mathbb{R}^2)}\|\partial_2 u\|_{L^1(\mathbb{R}^2)} \text{ for all } u \in C_c^\infty(\mathbb{R}^2).$$

(A similar inequality holds in higher dimensions.)

(ii) (Ladyzhenskaya's inequality) Deduce, by applying (i) to v^2, that

$$\|v\|^2_{L^4(\mathbb{R}^2)} \le \sqrt{2}\|v\|_{L^2(\mathbb{R}^2)}\|Dv\|_{L^2(\mathbb{R}^2)} \text{ for all } v \in C_c^\infty(\mathbb{R}^2).$$