C4.3 Functional Analytic Methods for PDEs - Sheet 4 of 4

Read Chapter 4 and prove the few statements whose proofs were left out as exercises. (Not to be handed in.)

Do:

Q1. Let X be a reflexive Banach space, $A \subset X$ be a weakly closed, non-empty subset, and $I : A \to \mathbb{R}$ be continuous. Suppose that the following conditions hold.

- I is weakly lower semi-continuous: for every sequence $(x_m) \subset A$ converging weakly to x it holds that
$$I(x) \leq \liminf_{m\to\infty} I(x_m).$$
- I is coercive: $I(x) \to \infty$ whenever $\|x\| \to \infty$, i.e. for every $N > 0$, there exists R such that $I(x) \geq N$ for every $x \in A$ with $\|x\| > R$.

Let
$$\alpha = \inf\{I(x) : x \in U\} \in [-\infty, \infty).$$
($\alpha < \infty$ as U is non-empty.)

Show that if $(x_m) \subset A$ is a minimizing sequence for I (i.e. $I(x_m) \to \alpha$), then (x_m) is bounded and hence has subsequence (x_{m_j}) which converges weakly to some $x \in A$. Deduce that x is a minimizer of I, i.e. $I(x) \leq I(z)$ for every $y \in A$.

Q2. Let X be a normed vector space.

(i) Let $(y_m) \subset X$ converges weakly to $y \in X$. By applying Mazur's theorem to the closed convex hull of (y_m), show that there is a sequence $m_j \to \infty$ and coefficients $c_1^{(j)}, \ldots, c_{m_j}^{(j)} \in [0, 1]$ such that $\sum_{j=1}^{m_j} c_i^{(j)} = 1$ and
$$\tilde{y}_j = \sum_{i=1}^{m_j} c_i^{(j)} y_i \text{ converges strongly to } y.$$

(ii) Let K be a closed, convex, non-empty subset of X and $I : K \to \mathbb{R}$ be continuous. Suppose that I is convex, i.e.
$$I((1-t)x_0 + tx_1) \leq (1-t)I(x_0) + tI(x_1) \text{ for all } x_0, x_1 \in K, t \in [0, 1]. \qquad (\dagger)$$
Show that I is weakly lower semi-continuous.

Q3. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $1 \le p \le \frac{n+2}{n-2}$, and consider, for a given $u_0 \in H^1(\Omega)$ the boundary value problem

$$\begin{cases} \Delta u &= |u|^{p-1}u \quad \text{in } \Omega, \\ u &= u_0 \qquad\quad \text{on } \partial\Omega. \end{cases} \tag{*}$$

(i) Formulate the notion of H^1 weak solution to the boundary value problem (*). You may need to use Gagliardo-Nirenberg-Sobolev's embedding theorem to justify the convergence of the integral corresponding to $|u|^{p-1}u$.

(ii) Using Gagliardo-Nirenberg-Sobolev's embedding theorem, show that the variational energy

$$I[u] = \int_\Omega \left[\frac{1}{2}|Du|^2 + \frac{1}{p+1}|u|^{p+1}\right]$$

is well-defined for $u \in H^1(\Omega)$. Furthermore, show that I is strictly convex (i.e. I is convex and the convexity inequality (†) is attained if and only if $x = y$ or $t = 0$ or $t = 1$).

(iii) Show that I has a unique minimizer in the set $X := \{u \in H^1(\Omega) : u - u_0 \in H_0^1(\Omega)\}$.

(iv) Differentiating under the integral sign, deduce that (*) has a weak solution. (*It is also true that the weak solution is unique, but this is harder.*)

Q4. Let Ω be a bounded domain in $\mathbb{R}^n$ and suppose $b_i \in L^\infty(\Omega)$ satisfies $\partial_i b_i \le 0$ in the weak sense, i.e.

$$\int_\Omega b_i \partial_i \varphi \ge 0 \text{ for all } \varphi \in C_c^\infty(\Omega).$$

Consider the operator

$$Lu = -\Delta u + b_i \partial_i u$$

and let B be the associated bilinear form. Show that B is coercive, namely there exists $C > 0$ such that $B(w, w) \ge C\|w\|^2_{L^2(\Omega)}$ for every $w \in C_c^\infty(\Omega)$.

Q5. Let Ω be a Lipschitz bounded domain of $\mathbb{R}^n$. Suppose that $a_{ij}, b_i, c \in L^\infty(\Omega)$, (a_{ij}) is uniformly elliptic and consider the operator

$$Lu = -\partial_i(a_{ij}\partial_j u) + b_i \partial_i u + cu.$$

Suppose that uniqueness holds for L: the only function in $H_0^1(\Omega)$ satisfying $Lu = 0$ in Ω in the weak sense is the trivial function. We knew that for every $f \in L^2(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ such that $Lu = f$ in Ω in the weak sense. We denote the map sending f to u as A so that $A : L^2(\Omega) \to H_0^1(\Omega)$ is linear. The aim of this exercise is to show the boundedness of A.

(i) Testing the equation $Lu = f$ against u, show that there exists $C_1 > 0$ independent of f such that

$$\|Af\|_{H^1(\Omega)} \leq C_1(\|f\|_{L^2(\Omega)} + \|Af\|_{L^2(\Omega)}) \text{ for all } f \in L^2(\Omega).$$

(ii) Use (i) to show that if A is unbounded, then there exist $f_m \in L^2(\Omega)$ and $u_m \in H_0^1(\Omega)$ such that $Lu_m = f_m$ and $1 = \|u_m\|_{L^2(\Omega)} \geq m\|f_m\|_{L^2(\Omega)}$. Show further that $f_m \to 0$ in $L^2(\Omega)$ and (u_m) is bounded in $H^1(\Omega)$.

(iii) With the same notation as in (ii) and using Rellich-Kondrachov's compactness theorem, show that there is a subsequence (u_{m_j}) which converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some function $u \in H_0^1(\Omega)$ which satisfies $Lu = 0$. Deduce a contraction from this convergence and conclude the argument.

Q6. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Define an operator $K : L^2(\Omega) \to L^2(\Omega)$ as follows. For every $f \in L^2(\Omega)$, we knew that there exists a unique $u \in H_0^1(\Omega)$ such that $-\Delta u = f$ in Ω in the weak sense. We assign $Kf = u$.

(i) Show that K is a compact bounded linear operator.

(ii) Let S be the restriction of K to $H_0^1(\Omega)$. Show that S is a compact bounded linear operator from $H_0^1(\Omega)$ into itself.