C4.1 Further Functional Analysis – Problem Sheet 2

For classes in Week 4 of MT

- This example sheet is based on the material up to section 6 of the notes together with Appendix A; the relevant video material is covered in the videos up to and including 7.1.
- Please send comments, corrections, clarifications to `stuart.white@maths.ox.ac.uk`.
- Please hand in the questions in Section B. You *may* also hand in the questions in Section A, or **exactly one** question from Section C (but not both).

1 Section A

1. Let X and Y be normed spaces and $T \in \mathcal{B}(X, Y)$
   (a) Show $(\mathrm{ran}\ T)^\circ = \ker T^*$.
   (b) Use the Hahn-Banach theorem to show $(\mathrm{ran}\ T^*)_\circ = \ker T$.
2. Let X be a normed space.
   (a) Let $C \subset X$ be convex set. Show that the closure, $\overline{C}$ is convex.
   (b) Given a subset $A \subset X$, show that
   $$\{\sum_{i=1}^{n} \lambda_i a_i : n \in \mathbb{N},\ a_i \in A,\ \lambda_i \geq 0,\ \sum_i \lambda_i = 1\}$$
   is the smallest convex subset of X containing A. This is known as the *convex hull* of A, and denoted $\mathrm{co}(A)$.
   (c) Let $C_1, C_2 \subseteq X$ be closed convex sets. Show that $\mathrm{co}(C_1 \cup C_2)$ is closed.

   > Mea culpa. This problem is not true — intituition for infinite dimensional convex geometry is misleading. Philip's proposed counter example works perfectly. Take two closed subspaces Y and Z of a normed space X whose sum $Y + Z$ is not closed as given on problem sheet 1. Then $\mathrm{co}(Y \cup Z) = Y + Z$ (containment from left to right is as $Y + Z$ is convex; but given any $y \in Y$ and $z \in Z$, we have $y + z = (2y + 2z)/2 \in \mathrm{co}(Y \cup Z)$.

   (d) Given a subset $A \subset X$, show that $\overline{\mathrm{co}}(A)$ is the smallest closed convex subset of X containing A. This is known as the closed convex hull of A, denoted $\overline{\mathrm{co}}(A)$.

   [You will perhaps find the closed convex hull construction, and in particular the version in (c) useful elsewhere on the example sheet. This cryptic remark is certainly less helpful than intended too – let's make sure to take the closed convex hull in 6c.]

2 Section B

3. Let X be a normed vector space and let Y be a subspace of X.
   (a) Suppose that Y is finite-dimensional. Show that Y is complemented in X, and that if Z is any closed subspace of X such that $X = Y \oplus Z$ algebraically, then X is in fact the topological direct sum of Y and Z.
   (b) What can you say if Y has finite codimension in X? [*Recall that the codimension of Y in X is the dimension of the quotient vector space X/Y.*]

4. (a) Let X be a Banach space and suppose that $\{x_n : n \geq 1\}$ is a bounded subset of X. Show that there exists a unique operator $T \in \mathcal{B}(\ell^1, X)$ such that $Te_n = x_n$ for all $n \geq 1$ and $\|T\| = \sup_{n \geq 1} \|x_n\|$.
   (b) Prove that if X is a separable Banach space then $X \cong \ell^1/Y$ for some closed subspace Y of ℓ^1.
   (c) Deduce that ℓ^1 contains closed subspaces which are uncomplemented.
   [*You may assume that any closed infinite-dimensional subspace of ℓ^1 has non-separable dual. We might prove this at the end of the course.*]

5. (a) Let X be an infinite dimensional real normed space, and $f : X \to \mathbb{R}$ a linear functional. Show that if there is an open ball $B_X^0(x_0, r)$ such that $f(x) > 0$ for $x \in B_X^0(x_0, r)$, then f is continuous. Deduce that if f is unbounded, then $\ker f$ is dense in X.
   (b) Use the previous result to show that any infinite dimensional normed space X can be decomposed into a union $A \cup B$ of disjoint convex sets, with both A and B dense in X.

6. (a) Let C be a convex absorbing subset of a normed space. Show
$$\{x \in X : p_C(x) < 1\} \subseteq C \subseteq \{x \in X : p_C(x) \leq 1\},$$
   with equality in the first inclusion when C is open, and equality in the second when C is closed.
   (b) Let C be a convex balanced subset of a normed space, which contains a neighbourhood of 0 and is bounded. Show that p_C gives an equivalent norm on X.
   (c) Let Y be a subspace of a normed space $(X, \cdot)$, and let $|||\cdot|||$ be an equivalent norm on Y. Show that $|||\cdot|||$ can be extended to an equivalent norm on X.

7. Let X and Y be normed vector spaces and let $T \in \mathcal{B}(X, Y)$. Suppose there exists a constant $r > 0$ such that $\|T^* f\| \geq r\|f\|$ for all $f \in Y^*$.
   (a) Using the Hahn-Banach Separation Theorem, or otherwise, show that $B_Y(r)$ is contained in the closure of $T(B_X)$.
   (b) If X is complete, deduce that T is a quotient operator, and that T is an isometric quotient operator if T^* is an isometry.

   [This question is asking you to complete the missing bits from Theorem 5.10.]

8. Let $X = \ell^\infty$ and $Sx = (x_{n+1})$ for $x = (x_n) \in X$. Moreover, let $T = I - S$.
   (a) Show that $\operatorname{Ker} T = \{(\lambda, \lambda, \lambda, \dots) : \lambda \in \mathbb{F}\}$ and that $\operatorname{Ran} T \cap \operatorname{Ker} T = \{0\}$.
   (b) Let $Y = \operatorname{Ran} T \oplus \operatorname{Ker} T$ and let $P\colon Y \to Y$ be the projection onto $\operatorname{Ker} T$ along $\operatorname{Ran} T$. By considering the operators
   $$A_n = \frac{1}{n}\sum_{k=0}^{n-1} S^k, \quad n \geq 1,$$
   or otherwise, show that P is bounded and that $\|P\| = 1$.
   (c) Prove that there exists a functional $f \in X^*$ with $\|f\| = 1$ such that $f(Sx) = f(x)$ for all $x \in X$ and
   $$f(x) = \lim_{n\to\infty} x_n$$
   whenever $x = (x_n) \in c$.[1] Evaluate $f(x)$ when x is a periodic sequence.

9. Let X be a normed vector space and let Y be a subspace of X.
   (a) Writing $Y^{\circ\circ} = (Y^\circ)^\circ$ for the double annihilator of Y in X^{**}, show that there exists an isometric isomorphism $T\colon Y^{**} \to Y^{\circ\circ}$ such that $T \circ J_Y = J_X|_Y$.
   (b) Show that if X is reflexive and Y is closed, then both Y and X/Y are reflexive.

3 Section C

This section consists of extensional exercises. While some, particularly on later sheets, might be (quite a lot) harder than the main exercises for the course, they won't all be. Some have been moved here compared to last year's sheets to keep the core sheet length under control (both for you and the TA!), and some are new. There is no requirement to do any of these exercises, and they're included for your enjoyment and to let you know what else is true.

It is unlikely we will have time to discuss any of these exercises in the classes; though if there is particular demand we will see what we can do. I will be happy to discuss these questions in office hours (with notice), but only I've after I've taken questions on the lectures and other more core questions. You may hand in exactly one section C question in for marking if you wish (and only if you do not hand in section A).

1. (a) Let X be a normed vector space and let $P \in \mathcal{B}(X^{***})$ be given by $P = J_{X^*}J_X^*$. Show that P is the projection onto $J_{X^*}(X^*)$ along $J_X(X)^\circ$ and that $\|P\| = 1$.
   (b) (i) Show that if $T \in \mathcal{B}(\ell^\infty)$ with $\|T\| = 1$ and $Te_n = e_n$, $n \geq 1$, then $T = I$.
   (ii) Deduce that there does not exist a projection of norm 1 from ℓ^∞ onto c_0.
   (iii) Prove that there is no normed vector space X such that $X^* \cong c_0$.

[1] Recall that c is the subspace of ℓ^∞ consisting of convergent sequences.

2. Given a normed vector space X, we say that X is injective [2] if whenever Y is a subspace of a normed vector space Z and $T \in \mathcal{B}(Y, X)$ there exists an operator $S \in \mathcal{B}(Z, X)$ such that $\|S\| = \|T\|$ and $S|_Y = T$.

   (a) (i) Show that ℓ^∞ is injective

   (ii) By proving first that any operator $T \in \mathcal{B}(\ell^\infty, c_0)$ such that $Te_n = e_n$, $n \geq 1$, must have norm $\|T\| \geq 2$, or otherwise, show that c_0 is not injective.

   (iii) Is c_0 complemented in c, and if so what can you say about the norm of a complementing projection?

   (b) Suppose that X is an injective normed vector space, and Y is a subspace of a normed vector space Z such that Y is isomorphic to X. Prove that Y is complemented in Z.

3. Let Y and Z be closed subspaces of a Banach space X and suppose that $X^* = Y^\circ \oplus Z^\circ$ as a topological direct sum. Show that $X = Y \oplus Z$ as a topological direct sum.

SAW MT20

[2] The terminology comes from category theory; X is an injective object in the category of normed spaces with contractive linear maps.