C4.1 Further Functional Analysis – Problem Sheet 4

For classes in Week 8 of MT

- This example sheet is based on the material up to section 13 of the notes together with Appendix A; the relevant video material is covered in the videos numbered up to 15.1.
- Please send comments, corrections, clarifications to `stuart.white@maths.ox.ac.uk`.
- Please hand in the questions in Section B. You *may* also hand in the questions in Section A, or **exactly one** question from Section C (but not both).

1 Section A

1. Let X and Y be normed spaces $T \in \mathcal{B}(X,Y)$. Fill in the details required to show that T is compact if and only if for every bounded sequence $(x_n)_{n=1}^{\infty}$, there is a subsequence $(x_{n_k})_k$ such that $(Tx_{n_k})_k$ converges.
2. Show that c_0 embeds isometrically into $\mathcal{K}(\ell^2)$. Deduce that $\mathcal{K}(\ell^2)$ is not reflexive.
3. This question aims to revise your knowledge of the spectrum of self-adjoint operators on a Hilbert space. If you've not seen it before, then the second and fourth parts probably won't be warm up exercises. Let X be a Hilbert space and $T \in \mathcal{B}(X)$.
   (a) Show that if T is self-adjoint, then eigenvectors of T corresponding to distinct eigenvalues are orthogonal.
   (b) Show that T is surjective if and only if the adjoint $T^\star$ is bounded below. Use this to show that if $\lambda \in \sigma(T)$ then there is a sequence $(x_n)_{n=1}^{\infty}$ in S_X such that $(Tx_n, x_n) \to \lambda$. [Mea cupla: the original version of this question was if and only if. This was very silly, and is obviously not true (it fails in 2 dimensions)!]
   (c) Deduce that the spectrum of a self-adjoint operator is contained in $\mathbb{R}$.
   (d) If T is self-adjoint show that $\|T\| = \sup_{x \in S_X} |(Tx,x)|$ and deduce that $r(T) = \|T\|$.

2 Section B

4. (a) Let X and Y be normed vector spaces and let $T \in \mathcal{B}(X,Y)$. We say that T is *completely continuous* if, for every weakly convergent sequence (x_n) in X, the sequence (Tx_n) is norm-convergent in Y.
   (i) Show that if T is compact then T is completely continuous.
   (ii) Prove that the converse of (i) holds if X is reflexive. [*You may, if you wish, assume in addition that X is separable.*]
   (iii) Exhibit an operator which is completely continuous but not compact.

(b) Let $1 < p < \infty$. Show that $\mathcal{B}(\ell^p, \ell^1) = \mathcal{K}(\ell^p, \ell^1)$. Is $\mathcal{B}(c_0, \ell^p) = \mathcal{K}(c_0, \ell^p)$?

5. Let X and Y be normed vector spaces and let $T \in \mathcal{B}(X, Y)$. We say that T is *weakly compact* if the weak closure of $T(B_X)$ is weakly compact.
   (a) Show that T is weakly compact if and only if $\operatorname{Ran} T^{**} \subseteq J_Y(Y)$.
   (b) Prove that if T is weakly compact then T^* is weakly compact, and that if Y is complete then the converse holds too.

6. Let $K \in L^2(\mathbb{R}^2)$ and consider the map T sending $x \in L^2(\mathbb{R})$ to the function Tx defined by
$$(Tx)(t) = \int_{\mathbb{R}} K(s,t)x(s)\,\mathrm{d}s$$
whenever $t \in \mathbb{R}$ is such that the integral exists.
   (a) Show that T is a well-defined element of $\mathcal{B}(L^2(\mathbb{R}))$ with $\|T\| \le \|K\|_{L^2(\mathbb{R}^2)}$.
   (b) Prove that T is compact. [*You may use the fact that indicator functions of bounded rectangles span a dense subspace of* $L^2(\mathbb{R}^2)$.]

7. Let X, Y be Banach spaces and suppose that $T \in \mathcal{B}(X, Y)$. Show that T is Fredholm if and only if T^* is and that, if both operators are Fredholm, then $\operatorname{ind} T + \operatorname{ind} T^* = 0$.

8. Let X, Y and Z be Banach spaces and let $S \in \mathcal{B}(Y, Z)$ and $T \in \mathcal{B}(X, Y)$.
   (a) Show that if S, T are both Fredholm then so is ST and $\operatorname{ind} ST = \operatorname{ind} S + \operatorname{ind} T$.
   (b) Suppose now that ST is Fredholm. Prove that S is Fredholm if and only if T is Fredholm. Give an example in which neither S nor T is Fredholm.
   (c) Show that if $X = Y = Z$ and $ST = TS$ then ST is Fredholm if and only if S and T are both Fredholm.

9. Let X be the complex Banach space ℓ^1 and consider the left-shift operator $T \in \mathcal{B}(X)$ given by $Tx = (x_{n+1})_{n\ge1}$ for $x = (x_n)_{n\ge1} \in X$. Moreover let $\Gamma = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.
   (a) Show that for $\lambda \in \mathbb{C}$ the operator $T - \lambda$ is Fredholm if and only if $\lambda \notin \Gamma$, and determine the index $\operatorname{ind}(T - \lambda)$ whenever it is defined.
   (b) Let p be a complex polynomial. Prove that $p(T)$ is Fredholm if and only if $p^{-1}(\{0\}) \cap \Gamma = \emptyset$ and that, if this condition is satisfied, then
$$\operatorname{ind} p(T) = \frac{1}{2\pi i} \oint_\Gamma \frac{p'(\lambda)}{p(\lambda)}\,\mathrm{d}\lambda.$$

10. Let X be a Banach space and let $\{x_n : n \ge 1\}$ be a Schauder basis for X with basis projections P_n, $n \ge 1$, and let
$$|||x||| = \sup\{\|P_n x\| : n \ge 1\}, \quad x \in X.$$
Prove that $|||\cdot|||$ defines a complete norm on X.

11. Let X be a separable Hilbert space. An operator $T \in \mathcal{B}(X)$ is a *Hilbert-Schmidt* operator if there is an orthonormal basis $(e_n)_{n=1}^\infty$ for X such that $\sum_{n=1}^\infty \|T(e_n)\|^2 < \infty$.

(a) Show that if $(e_n)_{n=1}^\infty$ and $(f_m)_{m=1}^\infty$ are orthonormal bases for X, then $\sum_m \|T(f_m)\|^2 = \sum_n \|T(e_n)\|^2$ for any $T \in \mathcal{B}(X)$.

(b) Show that every Hilbert-Schmidt operator is compact.

(c) Give a characterisation in terms of eigenvalues and multiplicities of when a compact self-adjoint operator is Hilbert-Schmidt.

12. Prove Theorem 14.4: if X is a Banach space with a Schauder basis, then every compact operator on X is a norm limit of finite rank operators. [1]

[See lectures for a hint]

3 Section C

This section consists of extensional exercises. While some, particularly on later sheets, might be (quite a lot) harder than the main exercises for the course, they won't all be. Some have been moved here compared to last year's sheets to keep the core sheet length under control (both for you and the TA!), and some are new. There is no requirement to do any of these exercises, and they're included for your enjoyment and to let you know what else is true.

It is unlikely we will have time to discuss any of these exercises in the classes; though if there is particular demand we will see what we can do. I will be happy to discuss these questions in office hours (with notice), but only I've after I've taken questions on the lectures and other more core questions. You may hand in exactly one section C question in for marking if you wish (and only if you do not hand in section A).

1. (a) Let X be a Banach space and suppose that $\{x_n : n \geq 1\} \subseteq X \setminus \{0\}$ spans a dense subspace of X. Prove that $\{x_n : n \geq 1\}$ is a Schauder basis for X if and only if there exists a constant $M > 0$ such that

$$\left\| \sum_{k=1}^{m} \lambda_k x_k \right\| \leq M \left\| \sum_{k=1}^{n} \lambda_k x_k \right\|$$

for all $n \geq m \geq 1$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$.

(b) Let X be a Banach space which admits a Schauder basis $\{x_n : n \geq 1\}$ with associated basis functionals $f_n \in X^*$, $n \geq 1$.

(i) Show that the set $\{f_n : n \geq 1\}$ is *basic*, which is to say that it forms a Schauder basis for its closed linear span.

(ii) Assuming that X^* admits a Schauder basis, is $\{f_n : n \geq 1\}$ necessarily a Schauder basis for X^*?

(c) (i) Let F be a finite dimensional subspace of a normed space X. Let $\epsilon > 0$ and fix an $\epsilon/2$ net $y_1, \ldots, y_k$ for the unit sphere S_F, and for each i choose $f_i \in S_{X^*}$ with $f_i(y_i) = 1$. Show that any $x \in S_X$ with $f_i(x) = 0$ for all i satisfies $\|y\| \leq (1 + \epsilon)\|y + \lambda x\|$ for all $y \in F$ and all $\lambda \in \mathbb{F}$.

[1] Additional exercise. Show that regardless of separability, every compact operator on a Hilbert space is a limit of finite rank operators.

(ii) Use the previous part repeatedly to show that any infinite dimensional Banach space contains a basic sequence.

2. Suppose that $K \in L^2(\mathbb{R}^2)$ is a complex-valued function such that $K(s,t) = \overline{K(t,s)}$ for all $s,t \in \mathbb{R}$, and let $\lambda \in \mathbb{C} \setminus \{0\}$. Given $y \in L^2(\mathbb{R})$ we wish to find $x \in L^2(\mathbb{R})$ such that, as an identity in $L^2(\mathbb{R})$, we have

$$\lambda x(t) - \int_{\mathbb{R}} K(s,t)x(s)\,\mathrm{d}s = y(t), \quad t \in \mathbb{R}.$$

Find criteria for existence and uniqueness of solutions $x \in L^2(\mathbb{R})$, and in the case where there is a unique solution for arbitrary $y \in L^2(\mathbb{R})$ express x as a series.

3. Let X be a Banach space with a Schauder basis $\{x_n : n \geq 1\}$ with associated basis projections P_n and basis functionals $f_n \in X^*$, $n \geq 1$.

   (a) Somewhat giving away the answer to the last part of the previous question, show that for each $n \in \mathbb{N}$,

   $$\|f|_{\overline{\mathrm{Span}\{x_i : i > n\}}} \leq \|f - P_n^* f\| \leq (1+K)\|f|_{\overline{\mathrm{Span}\{x_i : i > n\}}},$$

   where K is the basis constant. Deduce that $\overline{\mathrm{Span}}\{f_n : n \in \mathbb{N}\} = X^*$ if and only if for every $f \in X^*$,

   $$\|f|_{\overline{\mathrm{Span}\{e_i : i > n\}}}\| \to 0,$$

   as $n \to \infty$. [In this case we say that $\{x_n : n \geq 1\}$ is a *shrinking Schauder basis.*].

   (b) Let $f \in X^*$. Observe that $P_n^* f \to f$ weak*, and use this to deduce that if X is reflexive, then $\{x_n : n \geq 1\}$ is shrinking.

   (c) Suppose that $\{x_i : i \in \mathbb{N}\}$ is shrinking. Let Y be the space of all sequences $(a_n)_{n=1}^{\infty}$ equipped with $\|(a_n)\| = \sup_n \|\sum_{i=1}^n a_i x_i\|$. Verify that this is a norm on Y, and that $T : X^{**} \to Y$ given by $(T\phi) = (\phi(f_n)_{n=1}^{\infty})$ is an isomorphism. Show too that if the basis constant is 1, then T is isometric. [For context, think about what is going on with the canonical basis of c_0.].

   (d) Deduce that X is a reflexive space if and only if $\{x_i : i \in \mathbb{N}$ is shrinking and for all sequence of scalars $(a_n)_{n=1}^{\infty}$, $\sum_{i=1}^{\infty} a_i x_i$ converges whenever $\sup_n \|\sum_{i=1}^n a_i x_i\| < \infty$.

4. The *James space* is the space X consisting of all sequences of real numbers $(a_n)_{n=1}^{\infty}$ such that $a_n \to 0$ and

$$\|(a_n)\| = \sup_{k \geq 2} \sup_{n_1 < n_2 < \cdots < n_k} \Big(\sum_{i=1}^{k-1} (a_{n_i} - a_{n_{i+1}})^2\Big)^{1/2} < \infty.$$

   (a) Show that X is a Banach space, and that the elements e_n (which have a 1 in the n-th position and zeros elsewhere) form a Schauder basis for X with basis constant 1.

(b) Suppose, with the aim of reaching a contradiction, that $\{e_n : n \in \mathbb{N}\}$ is not shrinking in the sense of the previous question. Use 11(a), to find $f \in X^*$ with $\|f\| = 1$, $\epsilon > 0$, a real sequence $(a_n)_{n=1}^{\infty}$ and $p_1 < q_1 < p_2 < q_2 < \dots$ such that the elements $x_n = \sum_{i=p_n}^{q_n} a_i e_i \in X$ have $\|x_n\| = 1$ and $f(x_n) > \epsilon$ for all n. By considering

$$b_n = \begin{cases} a_n/n, & p_n \leq n \leq q_n \\ 0, & \text{otherwise} \end{cases},$$

or otherwise reach a contradiction, and deduce that $\{e_n : n \in \mathbb{N}\}$ is shrinking.

(c) Given a real sequence (a_n) such that $\sup_n \|\sum_{i=1}^{n} a_i e_i\| < \infty$, show that $\lim_{n\to\infty} a_n$ exists. Use question C.3 to deduce that $J_X(X)$ is has co-dimension 1 in X^{**}.

(d) Show that X is isomorphic to X^{**}.

(e) In C.1(b)(ii), your example probably had the property the dual space was not separable. Can you now give an example with a separable dual space?

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