C4.1 Further Functional Analysis — Guide to Videos

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This document provides a guide to the videos for the course, and what they cover in the lecture notes. Each video has as a small table of contents, and you can also click through the captions to the video to navigate the videos. Some very small slips of the tongue have been corrected in the captions, and small typos and clarifications are also noted here.

My thanks to Sergio Giron Pacheo for help with video post production, captions and proof watching. Remaining errors remain my responsibility!

1 Introduction

1.1 Introduction

- Discussion of what is functional analysis, and what's in the course. Brief discussion of books.

1.2 Prerequistes and background

- Discussion of background and prerequistes needed for the course.

1.3 Logistics and Example Sheets

- Discussion of what's available in the lectures, office hours
- Structure of example sheets.

1.4 Normed spaces

- Background on normed spaces, setting out notation, as in Section 1 of the notes.

2 Hamel Bases and unbounded functionals

2.1 Hamel Bases Exist

- Covers Def 2.1 - Remark 2.6 from notes.

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2.2 Hamel Bases Uncountable

- Brief discussion of Baire's category theorem (Thm A.1).
- Covers Proposition 2.7 - Example 2.9

2.3 Unbounded functionals exist

- Covers Proposition 2.10 to end of chapter.
- 3:31. Example 2.11. The notation here is correct but potentially confusing. f here is the functional $X \to \mathbb{F}$ we are defining, and x the element of X. So x represents a polynomial, we define the functional f by $f(x) = x'(1)$. The same issue arises at 9:17.
- 9:54. Out of consistency with the above, it would have been better to use x rather than f for a vector in X_0 in this example. However what is written on the slide is correct.

3 Direct sums and quotient spaces

3.1 Direct sums, complemented subpaces

- Covers section 3 of the notes.
- 18:30: as is often the case when I talk about the Banach spaces ℓ^p, the case $p = 2$ may be being implicitly excluded. So when I say that these are 'not isomorphic to Hilbert spaces' I mean that 'for p not equal to two, these are not isomorphic to Hilbert spaces'!

3.2 Quotient vector spaces

- Reviews quotient spaces and first isomorphism theorem for vector spaces, as at the beginning of chapter 4 of notes.

3.3 Quotient norms

- Covers definition of quotient norm (just before Prop 4.1), Prop 4.1 and Def 4.2.

3.4 The canonical quotient operator

- Covers Def 4.3 though Example 4.8.

4 Quotient Operators

4.1 Quotient Operators

- Covers Lemma 4.9 to Example 4.12.
- For some reason my camera cuts out in the last 20 secs of this video, but the sound is still audible and slides visible.

4.2 Characterising quotient operators

- Covers Theorem 4.13

4.3 Quotient operators from Banach spaces

- Covers Lemma 4.14, and remark 4.15.

5 The Hahn-Banach Theorem(s) I

5.1 Introduction and sublinear functionals

- Covers beginning of chapter 5 of notes, through to the end of Remark 5.5.

5.2 Real Hahn-Banach

- Covers 5.6, 5.7 and 5.8

5.3 Complex Hahn-Banach

- Covers 5.9 - 5.12.
- From around 13:45 onwards I say 5.2 quite a lot and mean 5.12. This should be clear from the slides and has been corrected in the lecture captions.

6 The Hahn-Banach Theorem(s) II

6.1 Hahn-Banach separation

- Def 5.13, Thm 5.14.

6.2 Duality between embedding and quotient operators

- Part 1 covers 5.15, statement of 5.16, the material in Remark 5.17, and Cor 5.18
- Part 2 covers parts of the proof of 5.16.

6.3 Closed Range Theorem

- Theorem 5.19.

7 Reflexivity and uniform convexity

7.1 Reflexivity

- Covers chapter 6 of the notes, except for the discussion of reflexivity of ℓ^p for $1 < p < \infty$

7.2 ℓ^p is reflexive

- Discussion of reflexivity of ℓ^p for $1 < p < \infty$

7.3 Uniform convexity

- Covers 7.1 - 7.5/

8 Smoothness and reflexivity of L^p

8.1 Smoothness

- Covers 7.6 and theorem 7.7

8.2 Outline of reflexivity of L^p

- Covers initial section of chapter 8, and Prop 8.1 (just for $p > 1$)

8.3 Uniform convexity of L^p

- Convex functions and Theorem 8.2

8.4 Reflexivity of L^p

- Theorems 8.3 and 8.4

9 The weak and weak*-topologies I

9.1 Weak topologies on vector spaces

- Motivates need for weaker topologies
- Topological background on bases, weakest topologies defined making functionals continuous
- Definition 9.1
- Proposition 9.7

9.2 The weak topology

- Definition 9.2 (the weak topology), and examples
- 9.3, 9.4, 9.5 (b) and Prop 9.9

10 The weak and weak*-topologies II

10.1 The weak*-topology

- Def 9.2 (weak*-top), and basic properties
- Prop 9.5(a)
- Relationship between weak and weak*-topologies on X^*, Cor 9.8.

10.2 The Banach-Alaoglu theorem

- Covers 9.10 - 9.13.

10.3 Weak* Hahn-Banach separation

- Covers 9.14-9.17

11 Norm compactness

11.1 Norm compactness

- 10.1 - 10.6
- 6:17 I manage to say 'M closure is relatively compact or precompact'. What I mean to say is 'We say that M is relatively compact or precompact if $\overline{M}$ is compact'. It is fixed in the captions.

11.2 The Arzela Ascoli Theorem

- Covers 10.7-10.10. 10.11 is off syllabus, so left in the notes but not in the lectures.

12 Compact operators

12.1 Intro to compact operators

- Covers 11.1 - 11.5 but not the proof of 11.3.

12.2 Properties of compact operators

- Proof of 11.3
- At the beginning of the video I say Proposition 1.3, and mean to say 11.3! This is fixed in the captions.

12.3 Schauder's Theorem

- Theorem 11.6.
- 2:45 I say k, l go to 0 but mean that they go to ∞!

13 Fredholm Operators

13.1 Intro to Fredholm operators

- Covers 12.1 - 12.4

13.2 Index is locally constant

- Covers 12.5

13.3 Fredholm Alternative

- Covers 12.7 and 12.8
- 5:58 I say 'Kernel of $T + S$ is finite dimensional as well' but write (and mean to say) 'Kernel of $T^* + S^*$ is finite dimensional as well'. Fixed in the errata.

14 Spectral theory for compact operators

14.1 Spectral theory of compact operators

- Covers 13.1-13.3

14.2 Spectral theorem for compact self-adjoints

- Covers 13.4-13.6

15 Schauder bases

- Covers 14.1-14.3, and examples 14.5(a) and (b)

15.1 Examples of Schauder bases

- States basis test (see exercises), and revisits examples 14.5(b) and covers examples 14.5(c) and states 14.5(d)
- 6:15 a slightly weird transition as I don't mention that the basis test is also used to prove the Haar basis works for $L^p[0, 1]$ (though I don't prove this anyway).
- 18:59 I say the terms from 0 to n and mean to say the terms from 0 to s.

15.2 Principle of small perturbations

- Covers 15.1 and 15.2.

A Baire's Theorem and its applications

- Covers appendix A.