C4.1 Further Functional Analysis – Problem Sheet 1

For classes in Week 2 of MT

This example sheet is based on the material in sections 2, 3 and 4 of the notes, together with Appendix A. Probably after the 4th lecture is not a bad time to look over the video associated to appendix A.

Please hand in the questions in Section B. You *may* also hand in the questions in Section A, or **exactly one** question from Section C (but not both).

1 Section A

1. Let X be a Banach spaces and Y a normed space. Let $T \in \mathcal{B}(X, Y)$ be such that there exists $\delta > 0$ such that $\|Tx\| \geq \delta \|x\|$ for all $x \in X$. Show that $\operatorname{Ran}(T)$ is complete, and hence closed.

2. Let X be a vector space and suppose that Y is a subspace of X.
   (a) By extending a Hamel basis for Y to X, construct a linear map $P \colon X \to X$ such that $P^2 = P$ and $\operatorname{Ran} P = Y$.
   (b) Deduce that Y is *algebraically complemented* in X, which is to say that there exists a further subspace Z of X such that every $x \in X$ can be expressed uniquely as $x = y + z$ with $y \in Y$ and $z \in Z$.
   (c) Is the subspace Z in part (b) uniquely determined by Y?

   [One can achieve the main point of this question: subspaces are algebraically complemented directly, by the same Hamel basis extension argument hinted at in (a).]

3. Let X be a vector space on which two norms $\|\cdot\|, |||\cdot|||$ are defined, and suppose that $\|x\| \leq C|||x|||$ for some constant $C > 0$ and all $x \in X$.
   (a) Show that if X is complete with respect to one of the two norms then it is complete with respect to the other if and only if the two norms are equivalent.
   (b) Give an example in which $(X, |||\cdot|||)$ is complete but $(X, \|\cdot\|)$ is not.

   [See Question 4 for examples with $(X, \|\cdot\|)$ complete but $(X, |||\cdot|||)$ is not.]

2 Section B

4. Let X be an infinite-dimensional normed space, and suppose that $\{x_\alpha : \alpha \in A\}$ is a Hamel basis for X and that $\|x_\alpha\| = 1$ for all $\alpha \in A$. Given a vector $x \in X$ which has the expansion $x = \sum_{\alpha \in A} \lambda_\alpha x_\alpha$ we let

$$|||x||| = \sum_{\alpha \in A} |\lambda_\alpha|.$$

   (a) Check that $|||\cdot|||$ defines a norm on X.
   (b) Now let X be a Banach space. Show that $(X, |||\cdot|||)$ is not separable.

(c) Deduce that in the Closed Graph Theorem the assumption that the codomain be complete cannot be omitted.

[Note that this provides loads of examples of Banach spaces X with norms $\|\cdot\|$ and $|||\cdot|||$ as in Question 3, in which $(X,\|\cdot\|)$ is complete but $(X,|||\cdot|||)$ is not.]

5. Let X be an infinite-dimensional Banach space with norm $\|\cdot\|$, and let $f\colon X\to\mathbb{F}$ be an unbounded linear functional. Given a vector $x_0\in X$ such that $f(x_0)=1$, consider the linear operator $T\colon X\to X$ defined by

$$Tx = x - 2f(x)x_0, \quad x\in X.$$

Show that $T^2=I$. Hence show that the map $|||\cdot|||\colon X\to[0,\infty)$ given, for $x\in X$, by $|||x|||=\|Tx\|$ defines a complete norm on X which is not equivalent to $\|\cdot\|$.

6. (a) Let X, Y and Z be vector spaces and suppose that $T\colon X\to Y$ and $S\colon X\to Z$ are linear maps. Show that there exists a linear map $\pi\colon Z\to Y$ such that $T=\pi\circ S$ if and only if $\operatorname{Ker} S\subseteq\operatorname{Ker} T$.

(b) Hence or otherwise show that if $n\in\mathbb{N}$ and if $f_1,\dots,f_n$ and f are linear functionals on a vector space X, then $f\in\operatorname{span}\{f_1,\dots,f_n\}$ if and only if

$$\bigcap_{k=1}^{n}\operatorname{Ker} f_k\subseteq\operatorname{Ker} f.$$

(c) Let F be a finite dimensional subspace of a normed vector space, and write F° for the annhilator of F, i.e. $F^\circ=\{f\in X^*: f(x)=0, x\in F\}$. Then form the preanhilator $(F^\circ)_\circ$ of F°, so $(F^\circ)_\circ=\{x\in X: f(x)=0,\ f\in F^\circ\}$. Show directly (so without using the Hahn-Banach theorem or any results from B4.1 on annhilators), that $F=(F^\circ)_\circ$.

[We will use part (b) of this question a lot later in the course, so make sure you keep this result handy.]

7. Let $Y,Z\subseteq\ell^2$ be given by

$$\begin{aligned} Y &= \big\{(y_n)\in\ell^2 : y_{2n}=0 \text{ for all } n\ge 1\big\},\\ Z &= \big\{(z_n)\in\ell^2 : z_{2n-1}=nz_{2n} \text{ for all } n\ge 1\big\}. \end{aligned}$$

(a) Show that Y and Z are closed subspaces of ℓ^2 and that $Y\cap Z=\{0\}$.

(b) Letting $X=Y\oplus Z$ denote the algebraic direct sum of Y and Z, prove that X is dense in ℓ^2 but that $X\neq\ell^2$, and deduce that X is not the topological direct sum of Y and Z.

(c) Let $P\colon X\to X$ be the linear map given by $P(y+z)=y$ for all $y\in Y$, $z\in Z$. Show directly that P is unbounded.

8. Let X be a normed vector space and let Y and Z be subspaces of X such that $X=Y\oplus Z$ algebraically. Show that if Y is closed, then X is the topological direct sum of Y and Z if and only if the restriction $\pi|_Z\colon Z\to X/Y$ of the canonical quotient map $\pi\colon X\to X/Y$ is an isomorphism.

9. Let Y and Z be closed subspaces of a Banach space X with $Y \cap Z = \{0\}$. Equip the algebraic direct sum $Y \oplus Z$ with the ℓ^1-norm: $|||y + z||| = \|y\| + \|z\|$.
   (a) Show that $|||\cdot|||$ is complete on $Y \oplus Z$.
   (b) Show that the following are equivalent:
      i. $|||\cdot|||$ is equivalent to the original norm on $Y \oplus Z$ (as a subspace of X);
      ii. $Y \oplus Z$ is closed in X;
      iii. Y is complemented by Z in $Y + Z$ (so $Y \oplus Z$ is a topological direct sum).
10. Prove that the Closed Graph Theorem, the Inverse Mapping Theorem and the Open Mapping Theorem are all equivalent.

    [We saw that OMT⇒IMT⇒CGT in Appendix A, and potentially in your earlier courses. You should avoid using any form of the axiom of choice (or even a countable version of the axiom of choice). Countable choice plays a role in the proof of all of these theorems, in that it is needed for the Baire category theorem.]

3 Section C

This section consists of extensional exercise. While some, particularly on later sheets, might be (quite a lot) harder than the main exercises for the course, many won't be and have been moved here compared to last year's sheets to keep the core sheet length under control (both for you and the TA!). There is no requirement to do any of these exercises, and they're included for your enjoyment and to let you know what else is true.

It is unlikely we will have time to discuss any of these exercises in the classes; though if there is particular demand we will see what we can do. I will be happy to discuss these questions in office hours (with notice), but only I've after I've taken questions on the lectures and other more core questions. You may hand in exactly one section C question in for marking if you wish (and only if you do not hand in section A).

1. Show that the Baire category theorem holds for locally compact Hausdorff topological spaces. That is, show that if X is locally compact and Hausdorff, and $(U_n)_{n=1}^{\infty}$ is a sequence of dense open subsets of X, then $\cap_{n=1}^{\infty} U_n$ is dense in X.

   [A topological space X is *locally compact* if for every $x \in X$, there exists a non-empty compact set K and open set U with $x \in U \subseteq K$. (i.e. K is a compact neighbourhood of x).]

2. Show, in the spirit of Question 10, that the principle of uniform boundedness is also equivalent to the open mapping theorem, inverse mapping theorem and closed graph theorem.

3. Let X and Y be Banach spaces and suppose that $T \in \mathcal{B}(X, Y)$ is such that $\operatorname{Ran} T$ has finite codimension in Y. Show that $\operatorname{Ran} T$ is closed.

   [Recall that the codimension of Y in X is the dimension of X/Y.]

SAW MT20