C4.1 Further Functional Analysis – Problem Sheet 3

For classes in Week 6 of MT

- This example sheet is based on the material up to section 9 of the notes together with Appendix A; the relevant video material is covered in the videos numbered up to 10. Videos numbered 9 and 10 will be available by Mon 9 Nov.
- Please send comments, corrections, clarifications to `stuart.white@maths.ox.ac.uk`.
- Please hand in the questions in Section B. You *may* also hand in the questions in Section A, or **exactly one** question from Section C (but not both).

1 Section A

Hopefully all of these are correct this week!

1. Let X be a normed space. Prove that a sequence $(x_n)_{n=1}^{\infty}$ in X converges weakly to $x \in X$ if and only if $f(x_n) \to f(x)$ as $n \to \infty$ for all $f \in X^*$.

2. Let X be a vector space, and $Y \subset X'$ a subspace. What conditions on Y ensure that the $\sigma(X, Y)$ topology is Hausdorff? Check that the weak topology on X and weak*-topology on X^* are Hausdorff. Which of these requires the Hahn-Banach theorem?

3. Let C be a convex subset of a vector space X, and $Y \subset X'$ a subspace. Show that the closure of C in the $\sigma(X, Y)$-topology is convex.

2 Section B

4. Let X be a real normed vector space and let $x_0 \in S_X$. For $x \in X$, define the functions $F_x, G_x \colon \mathbb{R} \to \mathbb{R}$ by

$$F_x(t) = t - \|tx_0 - x\| \quad \text{and} \quad G_x(t) = \|tx_0 + x\| - t.$$

   (a) Show, for each $x \in X$, that F_x is bounded above and non-decreasing and that G_x is bounded below and non-increasing. Show also that $F_x(s) \leq G_x(t)$ for all $x \in X$ and $s, t \in \mathbb{R}$.

   (b) Let the functions $a, b \colon X \to \mathbb{R}$ be defined by

$$a(x) = \lim_{t\to\infty} F_x(t) \quad \text{and} \quad b(x) = \lim_{t\to\infty} G_x(t).$$

   Given $x \in X$, show that there exists $f \in S_{X^*}$ such that $f(x_0) = 1$ and $f(x) = c$ if and only if $a(x) \leq c \leq b(x)$. [Set $Y = \operatorname{span}\{x_0, x\}$ and use the one-step extension argument in the proof of the real Hahn-Banach extension theorem.]

   (c) Deduce that the norm is smooth at x_0 if and only if there exists a unique $f \in S_{X^*}$ such that $f(x_0) = 1$.

5. Let X be a real normed vector space.
   (a) Using the previous question or otherwise, prove:
      i. that if the norm of X^* is strictly convex then the norm of X is smooth;
      ii. that if the norm of X^* is smooth then the norm of X is strictly convex;
   (b) Show that the converse statements to (ai) and (aii) hold if X is reflexive.

6. Let X be a normed vector space. This question asks you to finish the proof of Proposition 9.5, and now we have a bit more machinery than in lectures, I suggest doing it in a slightly different order, as you can use the Banach-Alaoglu theorem to speed up the proof (recall that every continuous bijection from a compact space into a Hausdorff space is a homeomorphism).
   (a) Show that if X is separable then the weak* topology on B_{X^*} is metrisable, i.e. complete the job started in lectures.
   (b) Deduce that if X^* is separable then the weak topology on B_X is metrisable.

7. Let $X = \ell^2$ and let $M = \{n^{1/2}e_n : n \geq 1\} \subseteq X$. Show that 0 lies in the weak closure of M but that there exists no sequence (x_n) with terms in M such that $x_n \to 0$ weakly as $n \to \infty$. What does this tell you about metrisability of the weak topology on the space X?

8. (a) Let X be a normed vector space and let C a convex subset of X. Use a Hahn-Banach theorem to show that C has the same closure with respect to the norm and weak topologies.[1]
   (b) Deduce that if $(x_n)_{n=1}^{\infty}$ is a sequence in a normed space converging weakly to $x \in X$, then there is a sequence of finite convex combinations of the x_n's converging in norm to x. Illustrate this result for $X = \ell^p$ with $1 < p < \infty$ and $e_n \in \ell^p$ the standard vector with 1 in the n-the position.

9. Let (x_n) be a sequence in ℓ^1 with terms $x_n = (x_n(j))_{j\geq 1}$, $n \geq 1$, and suppose that $x_n \to 0$ weakly as $n \to \infty$. Show that if $\|x_n\|_1 \not\to 0$ as $n \to \infty$ then there exist $\varepsilon > 0$ and a subsequence (x_{n_k}) such that $\|x_{n_k}\|_1 \geq 5\varepsilon$, $k \geq 1$, and moreover

$$\sum_{j=1}^{p_k-1} |x_{n_k}(j)| < \varepsilon \quad \text{and} \quad \sum_{j=q_k+1}^{\infty} |x_{n_k}(j)| < \varepsilon$$

   for positive integers p_k, q_k such that $p_k < q_k < p_{k+1}$, $k \geq 1$.

   Explain why this leads to a contradiction, and deduce that ℓ^1 has the Schur property.

   [We will use the Schur property of ℓ^1 a few times on the next example sheet.]

10. Let $T : X \to Y$ be a linear map between normed spaces. Show that T is norm continuous if and only if it is weakly continuous (i.e. it is continuous with respect to the weak topologies on X and Y).

[1] This is one of my personal favourite versions of the Hahn-Banach theorem. You might want to use it in the proof of C5(a)⇒(b).

3 Section C

This section consists of extensional exercises. While some, particularly on later sheets, might be (quite a lot) harder than the main exercises for the course, they won't all be. Some have been moved here compared to last year's sheets to keep the core sheet length under control (both for you and the TA!), and some are new. There is no requirement to do any of these exercises, and they're included for your enjoyment and to let you know what else is true.

It is unlikely we will have time to discuss any of these exercises in the classes; though if there is particular demand we will see what we can do. I will be happy to discuss these questions in office hours (with notice), but only I've after I've taken questions on the lectures and other more core questions. You may hand in exactly one section C question in for marking if you wish (and only if you do not hand in section A).

1. (a) Let (Ω, Σ, μ) be a finite measure space.
   (i) Given $f \in L^1(\Omega)^*$, show that there exists $y \in L^2(\Omega)$ such that
   $$f(x) = \int_\Omega x(t)y(t)\,\mathrm{d}\mu(t)$$
   for all $x \in L^2(\Omega)$.
   (ii) For $r > 0$ let $\Omega_r = \{t \in \Omega : |y(t)| \geq r\}$. By considering the functions
   $$x_r(t) = \frac{\overline{y(t)}}{|y(t)|}\mathbb{1}_{\Omega_r}(t), \quad t \in \Omega,$$
   show that $\mu(\Omega_r) = 0$ for $r > \|f\|$, so that $y \in L^\infty(\Omega)$ with $\|y\|_\infty \leq \|f\|$.
   (iii) Deduce that $L^1(\Omega)^* \cong L^\infty(\Omega)$.

   (b) Sketch an argument showing how this approach can be extended to prove that $L^1(\Omega)^* \cong L^\infty(\Omega)$ when (Ω, Σ, μ) is a σ-finite measure space.

2. Show that $L^1([0,1])$ does not have the Schur property so that ℓ^1 and $L^1([0,1])$ are not isomorphic. Banach spaces.[2]

3. Let X be a Banach space.
   (a) Suppose that X is non-reflexive and let $\phi \in X^{**} \setminus J_X(X)$. Show that $J_X^{**}(\phi) \neq J_{X^{**}}(\phi)$ but that
   $$(J_X^{**}\phi)(J_{X^*}f) = (J_{X^{**}}\phi)(J_{X^*}f) = \phi(f), \quad f \in X^*.$$
   Deduce that $\|J_X^{**}(\phi) + J_{X^{**}}(\phi)\| = 2\|\phi\|$.
   (b) Prove that if X^{****} is strictly convex then X must be reflexive.

[2] in contrast to $L^2([0,1])$ and ℓ^2 which are both separable infinite dimensional Hilbert spaces, so isometrically isomorphic, or ℓ^∞ and $L^\infty[0,1]$ which are isomorphic (as shown by Pelczynski) but not isometrically isomorphic.

4. (a) Let X be a normed vector space and suppose that M is a bounded subset of X. Denoting by N the weak* closure of $J_X(M)$ in X^{**}, show that $\operatorname{diam} N = \operatorname{diam} M$.

   (b) Let X be a Banach space and suppose that $\phi \in X^{**}$ has the property that for every $\varepsilon > 0$ there exists $M \subseteq X$ with $\operatorname{diam} M < \varepsilon$ such that ϕ is contained in the weak* closure of $J_X(M)$. Prove that $\phi \in J_X(X)$.

   (c) Let X be a uniformly convex normed vector space. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\operatorname{diam}\big\{x \in B_X : \operatorname{Re} f(x) > 1 - \delta\big\} < \varepsilon$$

   for all $f \in S_{X^*}$.

   (d) Prove that any uniformly convex Banach space is reflexive.

5. Let G be a countable discrete group and consider the Banach space $\ell^\infty(G) = \{f : G \to \mathbb{R} : \|f\|_\infty = \sup_{g\in G} |f(g)| < \infty\}$. We have an action of G on $\ell^\infty(G)$ given by $(g \cdot f)(h) = f(g^{-1}h)$ for $f \in \ell^\infty(G)$ and $g, h \in G$. One definition of amenability for G is that there exists $\mu \in \ell^\infty(G)^*$ such that $\|\mu\| = \mu(1) = 1$ (where by $1 \in \ell^\infty(G)$, we mean the constant function with value 1) $\mu(g\cdot f) = \mu(f)$ for all $f \in \ell^\infty(G)$ and $\mu(f) \geq 0$ when $f \geq 0$. Such a μ is called an *invariant mean* for G.

   We also have an action of G on $\ell^1(G)$, defined in exactly the same way as above, i.e. $(g \cdot \nu)(h) = \nu(g^{-1}h)$ for $\nu \in \ell^1(G)$ and $g, h \in G$. Prove that the following are equivalent:

   (a) G is amenable.

   (b) there exists a sequence $(\nu_n)_{n=1}^\infty$ of probability measures on G (i.e. elements of $\ell^1(G)$ with $\|\nu_n\|_1 = 1$ and $\nu_n(g) \geq 0$ for all $g \in G$) such that $\lim_{n\to\infty} \|g \cdot \nu_n - \nu_n\|_1 = 0$ for all $g \in G$.

   (c) For every finite subset S of G, and $\epsilon > 0$ there exists a finite subset F of G such that

$$\frac{|sF \triangle F|}{|F|} < \epsilon,$$

   for $s \in S$. (Here $A \triangle B$ denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of two sets, and $sF = \{sf : f \in F\}$.).

   In (c), the set F is called a *Følner* set for (F, ϵ). By exhibiting Følner sets or otherwise show that $\mathbb{Z}$ is amenable.

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