C4.1 Further Functional Analysis – Problem Sheet 4
Three Solutions

These are solutions to Q8, 10 and 12 - which were not fully covered in classes.

1 Question 8

Let X, Y and Z be Banach spaces and let $S \in \mathcal{B}(Y, Z)$ and $T \in \mathcal{B}(X, Y)$.

1. Show that if S, T are both Fredholm then so is ST and $\operatorname{ind} ST = \operatorname{ind} S + \operatorname{ind} T$.

   Before getting going on this question it's useful to make sure you're happy with the following facts about finite codimensional spaces:

   - If $U \subset V \subset X$ are subspaces, and U is finite codimensional in V and V finite codimensional in X, then U is finite codimensional in X.
   - If U, V are finite codimensional subspaces of X, then $U \cap V$ is also finite codimensional in X. (Check that $U \cap V$ is finite codimensional in U and use the previous fact).

   The approach taken below is in the spirit of the course, heavily using Theorem 12.4. Another — and arguably nicer — solution can be obtained via some short exact sequence arguments in the setting of vector spaces.

> If S and T are Fredholm then there exist closed finite-codimensional subspaces U and V of X and Y, respectively, such that T maps U isomorphically onto $\operatorname{Ran} T$ and S maps V isomorphically onto $\operatorname{Ran} S$. Thus $\dim X/U = \dim \operatorname{Ker} T$ and $\dim Y/V = \dim \operatorname{Ker} S$
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> We want to find a closed finite codimensional subspace M of X such that ST maps M isomorphically onto a closed finite codimensional subspace of Z. The plan is to start in the middle, defining $W = V \cap \operatorname{Ran} T$, a closed finite-codimensional subspace of Y.
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> Then the space $M = U \cap T^{-1}(W)$ is closed in X and satisfies $\dim U/M = \dim \operatorname{Ran} T/W$, so M has finite codimension in U and hence in X. Similarly, the space $N = S(W)$ is closed in Z and $\dim \operatorname{Ran} S/N = \dim V/W$, so N has finite codimension in $\operatorname{Ran} S$ and hence in Z. Since ST maps M isomorphically onto N we see that ST is Fredholm.
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> Finally we use the fact that $\operatorname{ind} ST$ can be computed as $\operatorname{ind} ST = \dim X/M - \dim Z/N$ (by Theorem 12.4). Then
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> $$\begin{aligned} \operatorname{ind} ST &= \dim X/M - \dim Z/N \\ &= \dim X/U + \dim \operatorname{Ran} T/W - \dim Z/\operatorname{Ran} S - \dim V/W \\ &= \dim \operatorname{Ker} T - \dim Y/\operatorname{Ran} T + \dim Y/W \\ &\qquad - \dim Z/\operatorname{Ran} S + \dim Y/V - \dim Y/W \\ &= \operatorname{ind} T + \operatorname{ind} S. \end{aligned}$$

2. Suppose now that ST is Fredholm. Prove that S is Fredholm if and only if T is Fredholm. Give an example in which neither S nor T is Fredholm.

Suppose that ST is Fredholm. Note that $\operatorname{Ker} T \subseteq \operatorname{Ker} ST$ and $\operatorname{Ran} ST \subseteq \operatorname{Ran} S$. Hence $\operatorname{Ker} T$ is finite-dimensional and $\operatorname{Ran} S$ has finite-codimension in Z, so T is Fredholm if and only if $\operatorname{Ran} T$ has finite codimension in Y and S is Fredholm if and only if $\operatorname{Ker} S$ is finite-dimensional.

Since ST is Fredholm there exists a closed finite-codimensional subspace U of X such that ST maps U isomorphically onto $\operatorname{Ran} ST$. Let $V = T(U)$. Then $V \cap \operatorname{Ker} S = \{0\}$. As U is finite codimension in X, $T(U)$ is finite codimension in $\operatorname{Ran}(T)$. So $\operatorname{Ran} T$ is of finite codimension in Y if and only if V is.

Since $\operatorname{Ran} ST = S(V)$ has finite codimension in Z there exists a finite-dimensional subspace W of Y such that $Y = \operatorname{Ker} S \oplus V \oplus W$. Hence $\dim Y/V < \infty$ if and only if $\dim \operatorname{Ker} S < \infty$, so T is Fredholm if and only if S is.

For a suitable example let $X = Y = Z = \ell^2$ and consider $Sx = (x_2, x_4, x_6, \dots)$ and $Tx = (0, x_1, 0, x_2, 0, \dots)$. Then ST is the identity operator, and in particular Fredholm, but $\operatorname{Ker} S$ is infinite-dimensional and $\operatorname{Ran} T$ has infinite codimension, so neither S nor T is Fredholm.

3. Show that if $X = Y = Z$ and $ST = TS$ then ST is Fredholm if and only if S and T are both Fredholm.

We know from part (a) that if S and T are both Fredholm then so is ST. Suppose $Q = ST = TS$ is Fredholm. Then $\operatorname{Ker} T \subseteq \operatorname{Ker} Q$ and $\operatorname{Ran} Q \subseteq \operatorname{Ran} T$, and similarly $\operatorname{Ker} S \subseteq \operatorname{Ker} Q$ and $\operatorname{Ran} Q \subseteq \operatorname{Ran} S$. So S, T are both Fredholm.

2 Question 10

Let X be a Banach space and let $\{x_n : n \geq 1\}$ be a Schauder basis for X with basis projections P_n, $n \geq 1$, and let

$$|||x||| = \sup\{\|P_n x\| : n \geq 1\}, \quad x \in X.$$

Prove that $|||\cdot|||$ defines a complete norm on X.

Before we start a quick word about context. This result is used in the proof of Theorem 14.3, so we should avoid using Theorem 14.3 to prove it (next year I'll make this explicit in the question)! So we should not use the fact that the basis projections are bounded (which was explicitly or implicitly done in many proofs).

If $x \in X$ satisfies $|||x||| = 0$, then $P_n x = 0$ for all $n \geq 1$ and hence $x = \lim_{n\to\infty} P_n x = 0$. The other norm axioms are easily seen to be satisfied, so $|||\cdot|||$ is indeed a norm. Let (y_k) be a $|||\cdot|||$-Cauchy sequence in X. For each $k \geq 1$ there exists a scalar sequence $(\lambda_n^{(k)})$ such that $y_k = \sum_{n=1}^\infty \lambda_n^{(k)} x_n$, $k \geq 1$. Then for $n \geq 1$ we have

$$|\lambda_n^{(k)} - \lambda_n^{(\ell)}| \|x_n\| = \|(P_n - P_{n-1})(y_k - y_\ell)\| \leq 2|||y_k - y_\ell|||, \quad k \geq \ell,$$

and hence the sequence $(\lambda_n^{(k)})$ is Cauchy for each $n \geq 1$, and therefore convergent. Let $\lambda_n = \lim_{k\to\infty} \lambda_n^{(k)}$, $n \geq 1$. Given $\varepsilon > 0$ we may find $K \geq 1$ such that $|||y_k - y_\ell||| < \varepsilon$ for $k, \ell \geq K$. For fixed $n > m$ we deduce that

$$\left\| \sum_{j=m+1}^{n} (\lambda_j^{(k)} - \lambda_j^{(\ell)}) x_j \right\| = \|(P_n - P_m)(y_k - y_\ell)\| \leq 2|||y_k - y_\ell||| < 2\varepsilon, \quad k, \ell \geq K.$$

Let $\ell \to \infty$ to get

$$\left\| \sum_{j=m+1}^{n} (\lambda_j^{(k)} - \lambda_j) x_j \right\| \leq 2\varepsilon, \quad k \geq K.$$

Since the series $\sum_{j=1}^\infty \lambda_j^{(K)} x_j$ is convergent there exists $N \geq 1$ such that

$$\left\| \sum_{j=m+1}^{n} \lambda_j x_j \right\| \leq 2\varepsilon + \left\| \sum_{j=m+1}^{n} \lambda_j^{(K)} x_j \right\| < 3\varepsilon, \quad m, n \geq N.$$

Hence the series $\sum_{n=1}^\infty \lambda_n x_n$ is $\|\cdot\|$-Cauchy and therefore convergent with limit y, say. Now by the Cauchy condition for (y_n) we know that for $k, \ell \geq K$ we have

$$\|P_n(y_k - y_\ell)\| = \left\| \sum_{j=1}^{n} (\lambda_j^{(k)} - \lambda_j^{(\ell)}) x_j \right\| < \varepsilon, \quad n \geq 1.$$

Letting $\ell \to \infty$ we see that $\|P_n(y_k - y)\| \leq \varepsilon$ for $k \geq K$ and $n \geq 1$. Hence $|||y_k - y||| \leq \varepsilon$ for $k \geq K$, so the norm $|||\cdot|||$ is complete.

3 Question 12

Prove Theorem 14.4: if X is a Banach space with a Schauder basis, then every compact operator on X is a norm limit of finite rank operators. [1]

Let $(P_n)_{n=1}^\infty$ be the sequence of basis projections corresponding to a Schauder basis, so that there exists some $M > 0$ such that $\|P_n\| \leq M$ and $P_n x \to x$ for all $x \in X$. Let $T \in \mathcal{K}(X)$. We claim that $P_n T \to T$ as $n \to \infty$ in operator norm, from which it follows that T is in the norm closure of the finite rank operators.

[1] Additional exercise. Show that regardless of separability, every compact operator on a Hilbert space is a limit of finite rank operators. My generousity does not extend to a solution of the additional exercise — beyond drawing your attention to the trick in Q4 I discussed in class

Suppose this fails, then there exists a bounded sequence $(x_n)_n$ and $\epsilon > 0$ such that $\|(P_nT - T)x_n\| \geq \epsilon$. Passing to a subsequence we may assume that $Tx_n \to y \in X$ say. But then

$$\|P_nTx_n - y\| \leq \|P_n(Tx_n - y)\| + \|P_ny - y\| \leq M\|Tx_n - y\| + \|P_ny - y\| \to 0,$$

as $n \to \infty$. This contradiction proves the result.