Prelims Analysis III

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Trinity Term 2022

Definition of the integral of a function

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The fact that integration and differentation are 'inverses' will become a theorem called the Fundamental Theorem of Calculus (that needs some extra hypotheses!)

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Not every function will be integrable!

But once we've defined integration, we'll prove that every continuous function on a closed bounded interval is integrable.

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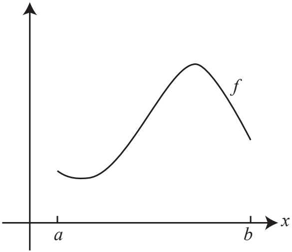

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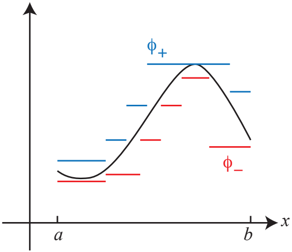

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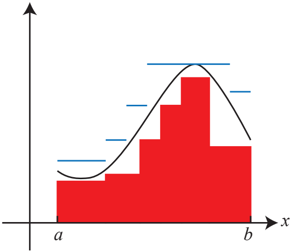

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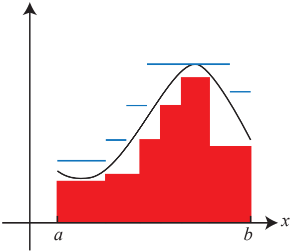

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Then we'll consider all steps functions $\phi_- \leq f$ and all step functions $\phi_+ \geq f$. We'll say that f is integrable if

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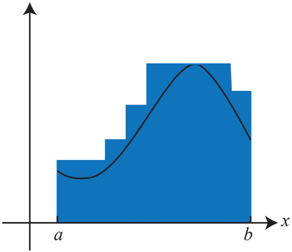

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Chapter 1A: The definition of integration

Step functions

Definition. Let $[a, b]$ be an interval. A function $\phi : [a, b] \to \mathbb{R}$ is called a step function if there is a finite sequence $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ such that ϕ is constant on each open interval (x_{i-1}, x_i).

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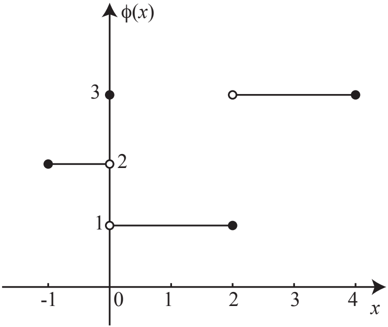
A partition $\mathcal{P}'$ given by $a = x'_0 \leq \cdots \leq x'_{n'} \leq b$ is refinement of $\mathcal{P}$ if every x_i is an x'_j for some j.

An example



$$\phi(x) = \begin{cases} 2 & \text{if } -1 \leq x < 0; \\ 3 & \text{if } x = 0; \\ 1 & \text{if } 0 < x \leq 2; \\ 3 & \text{if } 2 < x \leq 4. \end{cases}$$

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2. If $\mathcal{P}_1, \mathcal{P}_2$ are two partitions then there is a common refinement of both of them.
3. If ϕ_1, ϕ_2 are step functions then so are $\max(\phi_1, \phi_2)$, $\phi_1 + \phi_2$ and $\lambda\phi_i$ for any scalar λ.

Indicator functions

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Proof. Suppose first that ϕ is a step function adapted to some partition $\mathcal{P}$, $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$. Then ϕ can be written as a weighted sum of the functions $\mathbf{1}_{(x_{i-1}, x_i)}$ (each an indicator function of an open interval) and the functions $\mathbf{1}_{\{x_i\}}$ (each an indicator function of a closed interval containing a single point).

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In particular, the step functions on $[a, b]$ form a vector space, which we occasionally denote by $\mathscr{L}_{\mathrm{step}}[a, b]$.

I of a step function

Definition. Let ϕ be a step function adapted to some partition $\mathcal{P}$, and suppose that $\phi(x) = c_i$ on the interval (x_{i-1}, x_i). Then we define

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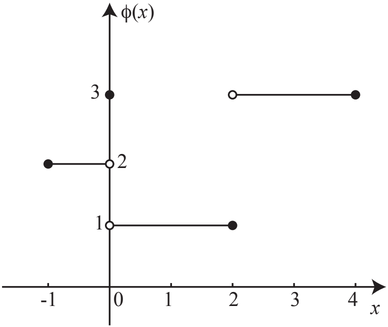
We call this $I(\phi)$ rather than $\int_a^b \phi$, because we are going to define $\int_a^b f$ for a class of functions f much more general than step functions. It will then be a theorem that $I(\phi) = \int_a^b \phi$, rather than simply a definition.

An example



$$I(\phi) = (2 \times 1) + (1 \times 2) + (3 \times 2) = 10.$$

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Now if ϕ is a step function adapted to both $\mathcal{P}_1$ and $\mathcal{P}_2$ then they have a common refinement $\mathcal{P}'$ and so

$$I(\phi; \mathcal{P}_1) = I(\phi; \mathcal{P}') = I(\phi; \mathcal{P}_2).$$

Linearity of I

Lemma 1.6. The map $I : \mathscr{L}_{\text{step}}[a, b] \to \mathbb{R}$ is linear:
$I(\lambda\phi_1 + \mu\phi_2) = \lambda I(\phi_1) + \mu I(\phi_2)$.

Majorants and minorants

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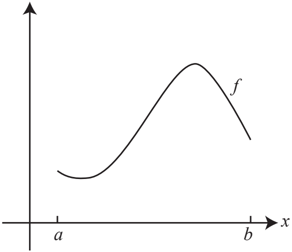

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Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

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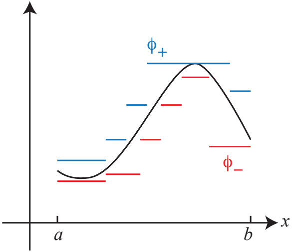

Definition of the integral

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We note that the sup and inf exist for any bounded function f.

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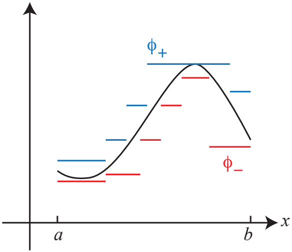
$$I(\phi_-) \leq I(\phi_+).$$



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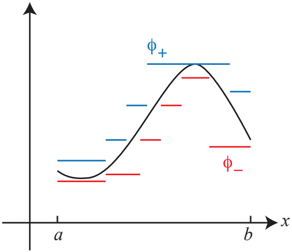

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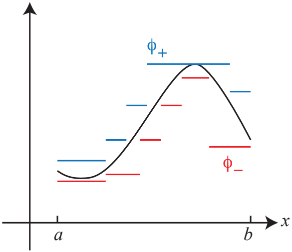

Hence, it is always the case that

$$\sup_{\phi_-} I(\phi_-) \leq \inf_{\phi_+} I(\phi_+).$$

It follows that when f is integrable, then

$$I(\phi_-) \leq \int_a^b f \leq I(\phi_+)$$

whenever $\phi_- \leq f \leq \phi_+$ are a minorant and majorant.

Minor remarks

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2. Integrals are often written using the dx notation. For example, $\int_0^1 x^2 dx$. This means the same as $\int_0^1 f$, where $f(x) = x^2$.

An important lemma

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(i) f is integrable;

(ii) for every $\epsilon > 0$, there is a majorant ϕ_+ and a minorant ϕ_- for f such that $I(\phi_+) - I(\phi_-) < \epsilon$.

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Then by the approximation property for sup and inf, there is a minorant ϕ_- such that

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Since the sup and inf are assumed to be equal, we deduce that

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Therefore, $\inf_{\phi_+} I(\phi_+)$ is squeezed between $\sup_{\phi_-} I(\phi_-)$ and $\sup_{\phi_-} I(\phi_-) + \epsilon$.
Since $\epsilon > 0$ was arbitrary, we deduce that inf and sup must be equal. In other words, f is integrable.

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So, $\int_a^b f$ is squeezed between $I(\phi_-)$ and $I(\phi_+)$, which differ by less than ϵ.

An example

Example The function $f(x) = x$ is integrable on $[0, 1]$, and

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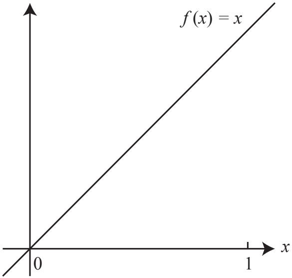

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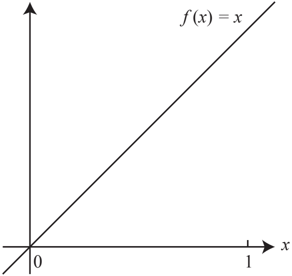

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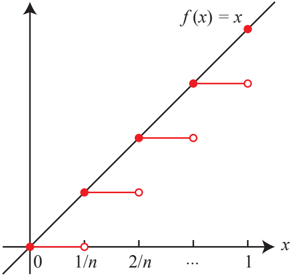

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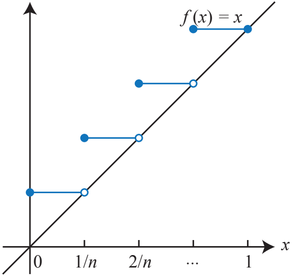

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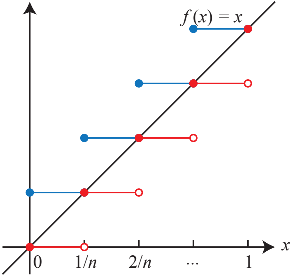

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$$I(\phi_-) = \sum_{i=0}^{n-1} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{2}(1 - \frac{1}{n})$$

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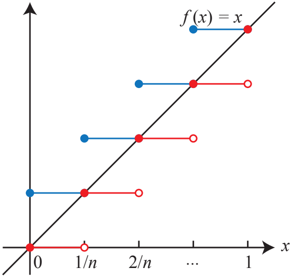

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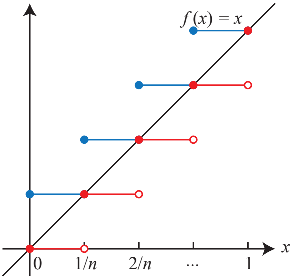

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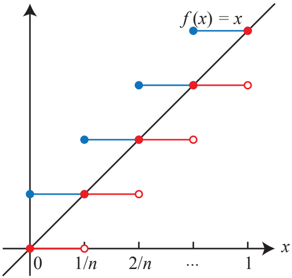

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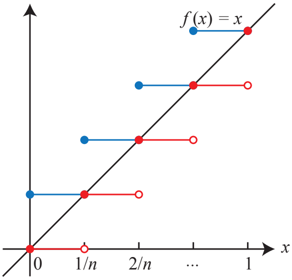

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Proof. Take $\phi_- = \phi_+ = \phi$, and the result is immediate. □

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Example The function $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

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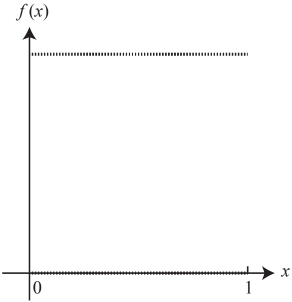

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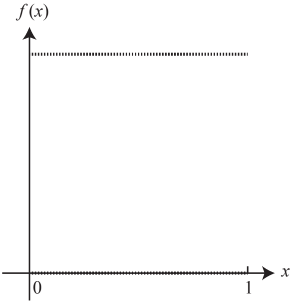

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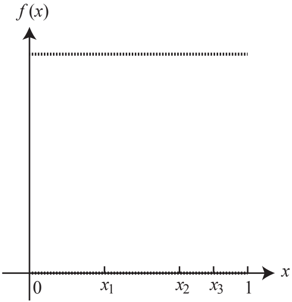

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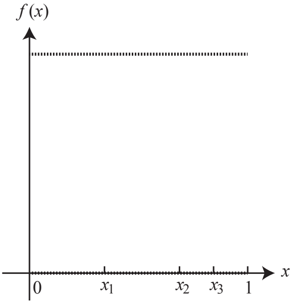

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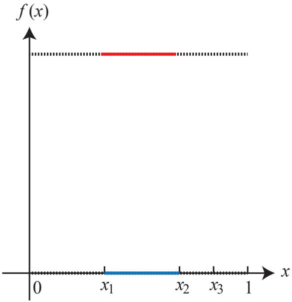

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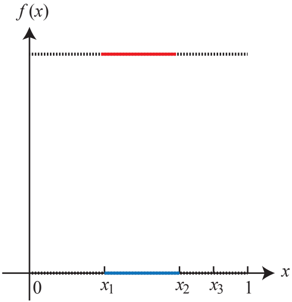

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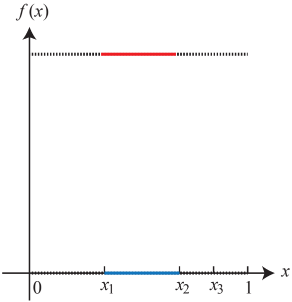

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So, f is not integrable.

Chapter 1B: Basic theorems about the integral

Monotonicity of the integral

Proposition 1.18(ii). If $f, g : [a, b] \to \mathbb{R}$ are integrable and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

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$$\int_a^b f = \sup_{\phi_-} I(\phi_-)$$

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But any minorant ϕ_- for f is a minorant for g. $\square$

Restricting to a subinterval

Proposition 1.13. Suppose that f is integrable on $[a, b]$. Then, for any c with $a < c < b$, f is Riemann integrable on $[a, c]$ and on $[c, b]$. Moreover $\int_a^b f = \int_c^b f + \int_a^c f$.

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Corollary 1.14. Suppose that $f : [a, b] \to \mathbb{R}$ is integrable, and that $[c, d] \subset [a, b]$. Then f is integrable on $[c, d]$.

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In this proof it is convenient to assume that

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Now observe that a minorant ϕ_- of f on $[a, b]$ is precisely the same thing as a minorant $\phi_-^{(1)}$ of f on $[a, c]$ juxtaposed with a minorant $\phi_-^{(2)}$ of f on $[c, b]$, and that $I(\phi_-) = I(\phi_-^{(1)}) + I(\phi_-^{(2)})$. A similar comment applies to majorants.

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$$\sup_{\phi_-} I(\phi_-) = \sup_{\phi_-^{(1)}} I(\phi_-^{(1)}) + \sup_{\phi_-^{(2)}} I(\phi_-^{(2)})$$

$$\inf_{\phi_+} I(\phi_+) = \inf_{\phi_+^{(1)}} I(\phi_+^{(1)}) + \inf_{\phi_+^{(2)}} I(\phi_+^{(2)}).$$

Proof (continued)

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Thus f is indeed integrable on $[a, c]$ and on $[c, b]$, and $\int_a^b f = \int_a^c f + \int_c^b f$. □

Linearity of the integral

Proposition 1.15. If f, g are integrable on $[a, b]$ then so is $\lambda f + \mu g$ for any $\lambda, \mu \in \mathbb{R}$. Moreover

$$\int_a^b (\lambda f + \mu g) = \lambda \int_a^b f + \mu \int_a^b g.$$

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Proof. This follows from two simpler claims:

1. λf is integrable and $\int_a^b \lambda f = \lambda \int_a^b f$
2. $f + g$ is integrable and $\int_a^b f + g = \int_a^b f + \int_a^b g$.

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Since $\epsilon > 0$ was arbitrary, we deduce that λf is integrable. Also,

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Finally, $\lambda = 0$ is easy because λf is then a step function, and its integral is 0.

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Hence, $\phi_- + \psi_-$ and $\phi_+ + \psi_+$ are minorants and majorants for $f+g$ satisfying

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Proof (continued)

We now want to show that $f+g$ is integrable and

$$\int_a^b f+g = \int_a^b f + \int_a^b g.$$

Let $\epsilon > 0$. We know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/2.$$

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□

Changing a function at finitely many points

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Corollary 1.16. If f is integrable on $[a, b]$, and if $\tilde{f}$ differs from f in finitely many points, then $\tilde{f}$ is also integrable.

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$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$.
By Proposition 1.10, $\tilde{f} - f$ is integrable.
Hence so is $\tilde{f} = (\tilde{f} - f) + f$, by Proposition 1.15. □

The maximum and minimum of integrable functions

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Proposition 1.17. Suppose that f and g are integrable on $[a, b]$. Then $\max(f, g)$ and $\min(f, g)$ are both Riemann integrable, as is $|f|$.

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$$\begin{aligned} \max(f, g) &= g + \max(f - g, 0) \\ \min(h, 0) &= -\max(-h, 0) \\ |h| &= \max(h, 0) - \min(h, 0). \end{aligned}$$

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Using these relations and Proposition 1.15, it is enough to prove that if f is integrable on $[a, b]$, then so is $\max(f, 0)$.

Proof (continued)

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as can be established by an easy case-check, according to the signs of x, y.

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It follows that if $\phi_- \leq f \leq \phi_+$ are minorant and majorant for f then

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$$I(\max(\phi_+, 0)) - I(\max(\phi_-, 0)) \leq I(\phi_+) - I(\phi_-).$$

Since f is integrable, this can be made arbitrarily small. □

Bounds on the integral

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(i) The constant function $\phi_-(x) = \inf_{x\in[a,b]} f(x)$ is a minorant for f on $[a, b]$, whilst $\phi_+(x) = \sup_{x\in[a,b]} f(x)$ is a majorant. Thus

$$(b-a)\inf_{x\in[a,b]} f(x) = I(\phi_-) \le \sup_{\phi_-} I(\phi_-) \le \int_a^b f,$$

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(iii) Apply (ii) to f and $|f|$, and also to $-f$ and $|f|$, obtaining $\pm\int_a^b f \leq \int_a^b |f|$.

The product of two integrable functions

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Proposition 1.19. Suppose that $f, g : [a, b] \to \mathbb{R}$ are two integrable functions. Then their product fg is integrable.

Proof. Write $f = f_+ - f_-$, where $f_+ = \max(f, 0)$ and $f_- = -\min(f, 0)$, and similarly for g.

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Suppose, then, that $f, g \geq 0$.

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Replacing ϕ_- with $\max(\phi_-, 0)$ if necessary (and similarly for ψ_-), we may assume that $\phi_-, \psi_- \geq 0$ pointwise.

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Replacing ϕ_+ with $\min(\phi_+, M)$, where
$M = \max\{\sup_{[a,b]} f, \sup_{[a,b]} g\}$ (and similarly for ψ_+) we may assume that $\phi_+, \psi_+ \leq M$ pointwise.

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By refining partitions if necessary, we may assume that all of these step functions are adapted to the same partition $\mathcal{P}$.
Now observe that $\phi_-\psi_-, \phi_+\psi_+$ are both step functions and that $\phi_-\psi_- \leq fg \leq \phi_+\psi_+$ pointwise.

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Moreover, if $0 \leq u, v, u', v' \leq M$ and $u \leq u'$, $v \leq v'$ then we have

$$u'v' - uv = (u' - u)v' + (v' - v)u \leq M(u' - u + v' - v).$$

Applying this on each interval of the partition $\mathcal{P}$, with $u = \phi_-$, $u' = \phi_+$, $v = \psi_-$, $v' = \psi_+$, we have

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Since $\varepsilon > 0$ was arbitrary, the result follows. □

Chapter 2A: Integrating a continuous function

Continuous functions are integrable

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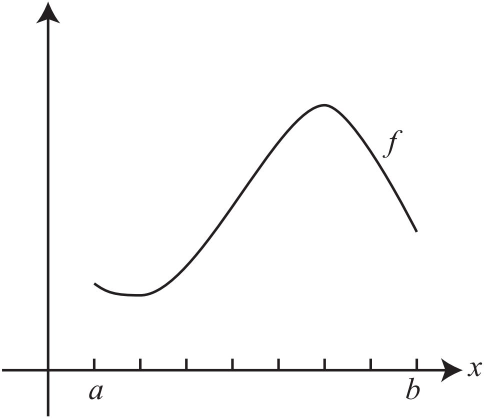

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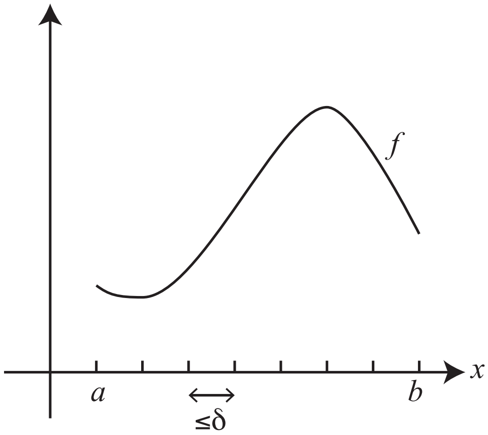

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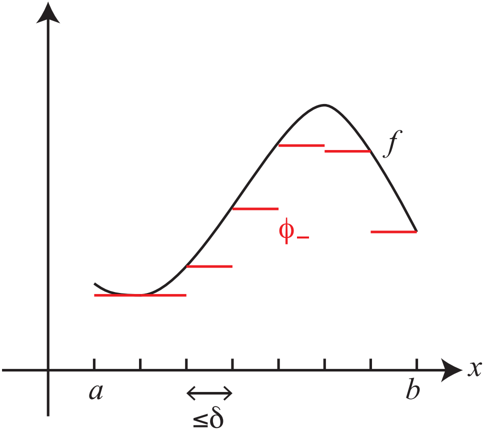

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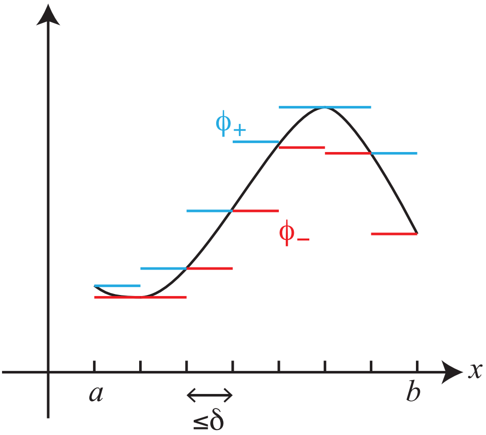

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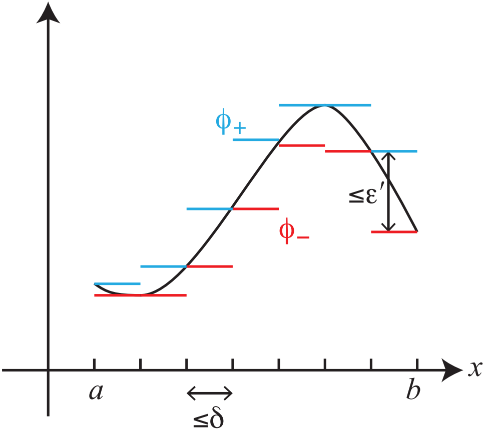

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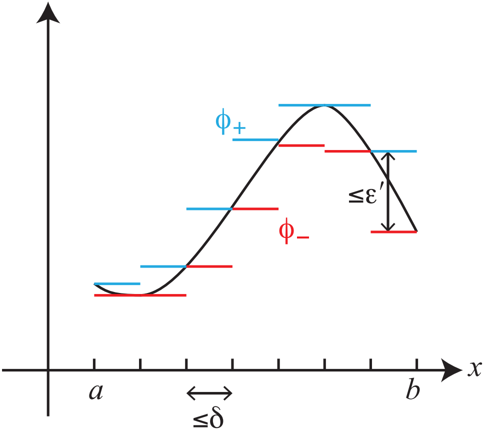

$$I(\phi_+) - I(\phi_-) \leq (b - a)\epsilon'.$$

Proof

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We want to show that, for any $\epsilon > 0$, there is a minorant ϕ_- and a majorant ϕ_+ such that $I(\phi_+) - I(\phi_-) < \epsilon$.

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We'll set $\epsilon' = \epsilon/(b - a)$.

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Proof (continued)

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Therefore $\phi_+(x) - \phi_-(x) < \epsilon'$ for all $x \in [a, b]$, including the points x_i.

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It follows that $I(\phi_+) - I(\phi_-) < \epsilon'(b - a) = \epsilon$. $\square$

Continuity on an open interval

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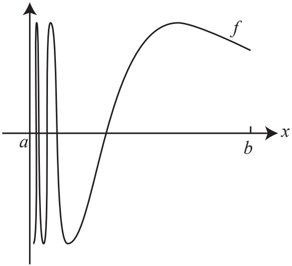

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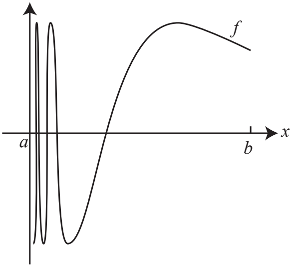

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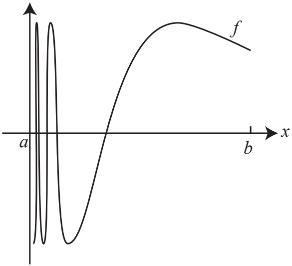

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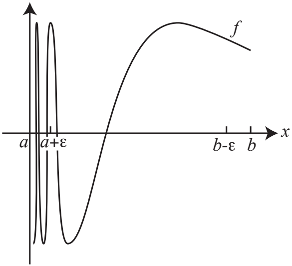

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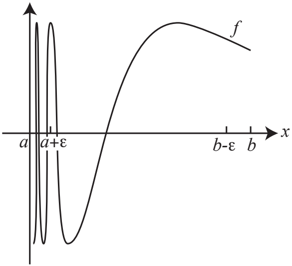

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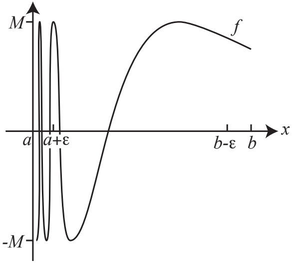

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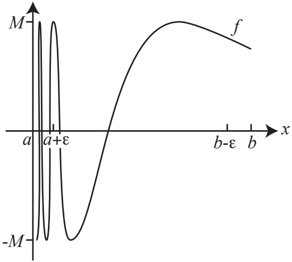

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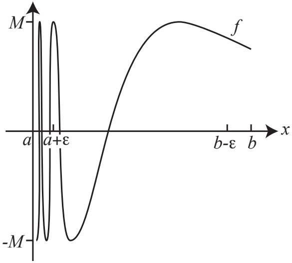

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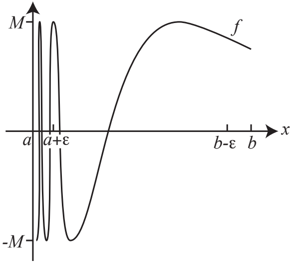

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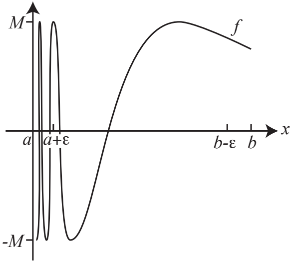

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Since $\epsilon > 0$ was arbitrary, f is integrable. $\square$

Integrating a non-negative continuous function

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Lemma 2.3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function with $f \geq 0$ pointwise and $\int_a^b f = 0$. Then $f(x) = 0$ for $x \in [a, b]$.

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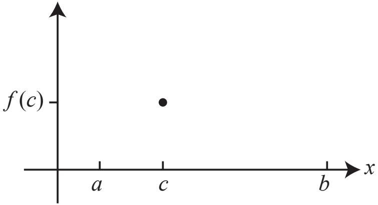

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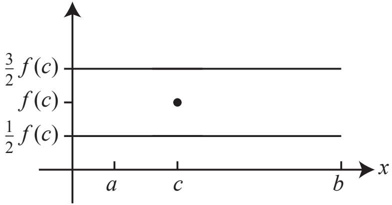

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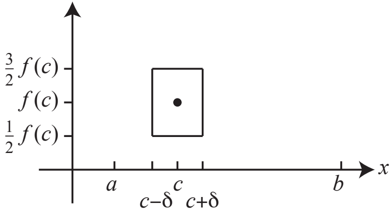

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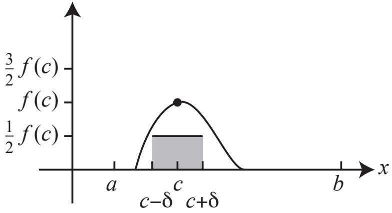

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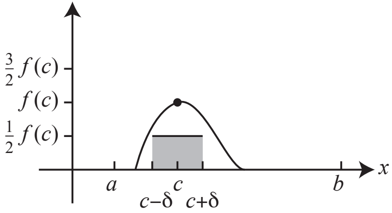

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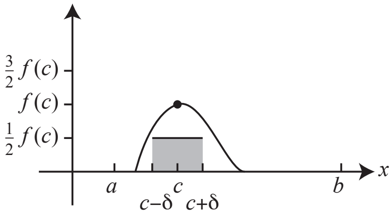

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Chapter 2B: Mean values, monotone functions

A first mean value theorem

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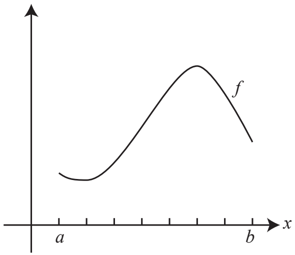

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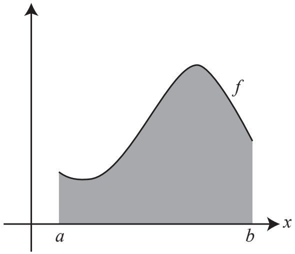

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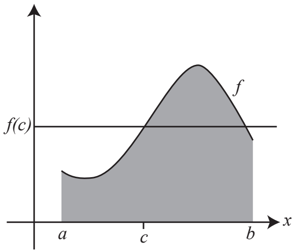

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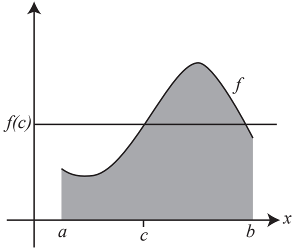

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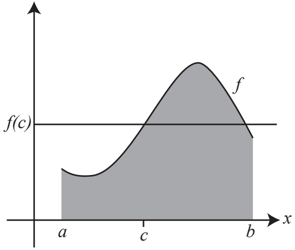

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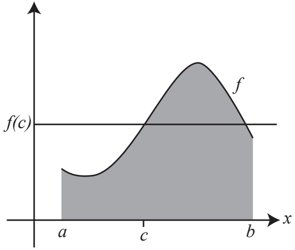

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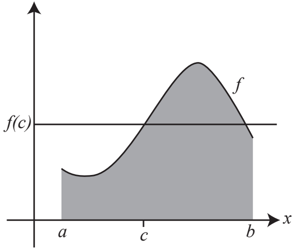

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By the intermediate value theorem, f attains every value in $[m, M]$, and in particular there is some c such that $f(c) = \frac{1}{b-a}\int_a^b f$. □

A second mean value theorem

Proposition 2.5. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and that $w : [a, b] \to \mathbb{R}$ is a nonnegative integrable function. Then there is some $c \in [a, b]$ such that

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$m \le \dfrac{\int_a^b fw}{\int_a^b w} \le M$. So, by IVT, there is a $c \in [a, b]$ s.t. $f(c) = \dfrac{\int_a^b fw}{\int_a^b w}$.

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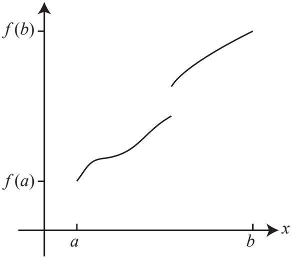

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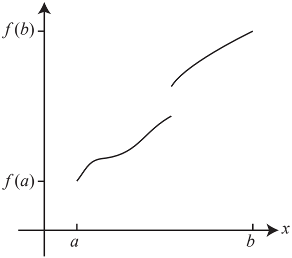

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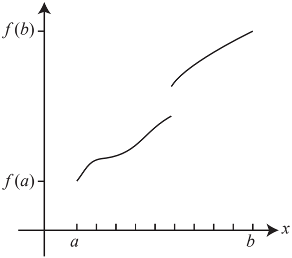

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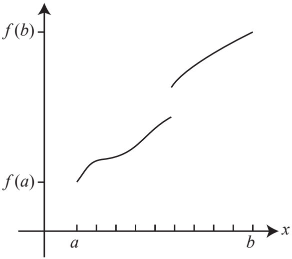
Let n be a positive integer, and consider the partition $\mathcal{P}$ of $[a, b]$ into n equal parts:

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b.$$

Proof (continued)



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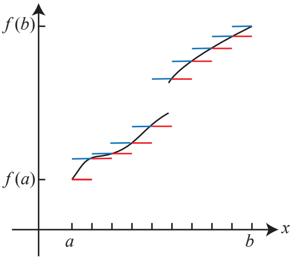
On (x_{i-1}, x_i), define $\phi_+(x) = f(x_i)$ and $\phi_-(x) = f(x_{i-1})$.



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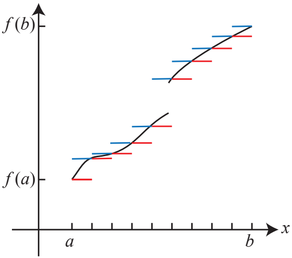

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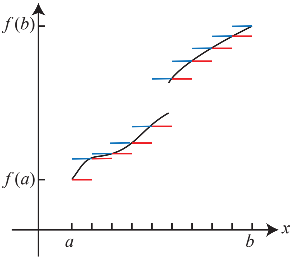

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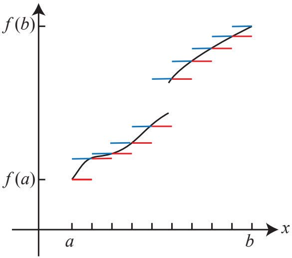

$$I(\phi_+) - I(\phi_-) = \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))(x_i - x_{i-1})$$
$$= \frac{b-a}{n}\sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))$$
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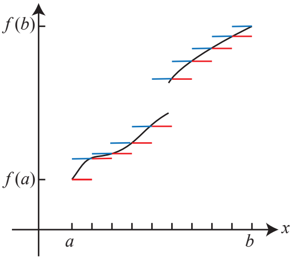

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Taking n large, this can be made as small as desired. □

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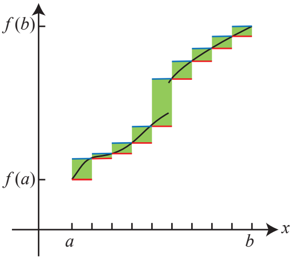

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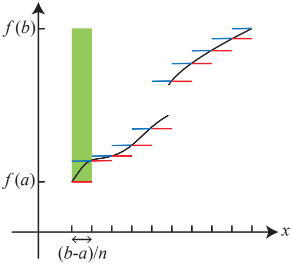

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Chapter 3A: Riemann sums

Riemann sums

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If $\mathcal{P}$ is a partition and $f : [a, b] \to \mathbb{R}$ is a function then by a Riemann sum adapted to $\mathcal{P}$ we mean an expression of the form

$$\Sigma(f; \mathcal{P}, \vec{\xi}) = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}),$$

where $\vec{\xi} = (\xi_1, \ldots, \xi_n)$ and $\xi_j \in [x_{j-1}, x_j]$.

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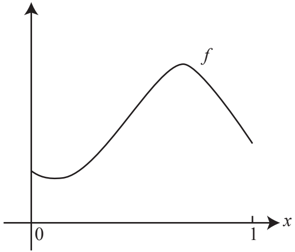

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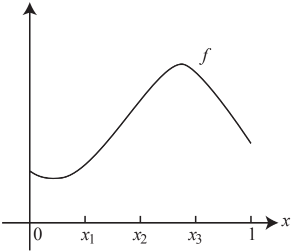

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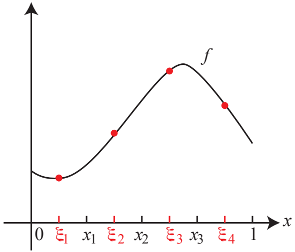

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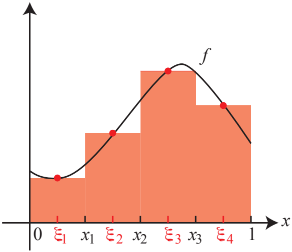

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Take $\xi_j^{(i)} = \frac{j}{i}$; then the Riemann sum $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ is equal to

$$S_i(f) := \frac{1}{i} \sum_{j=1}^{i} f(\frac{j}{i}).$$

By Proposition 3.2, if f is integrable then

$$S_i(f) \to \int_a^b f.$$

However, the converse is not true.

Further remarks

It is important that the limit must exist for any choice of $\vec{\xi}^{(i)}$.

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$$S_i(f) = 1 \qquad \text{for all } i.$$

Chapter 3B: Riemann sums (proofs)

Riemann sums and the integral

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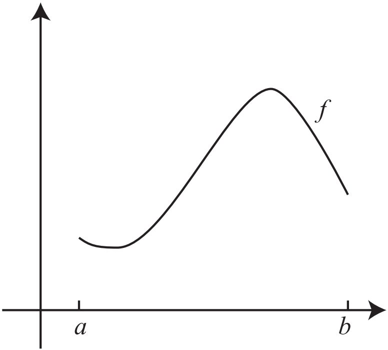

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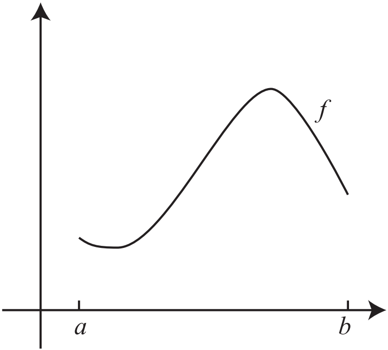

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We will show that there is a majorant ϕ_+ and minorant ϕ_- such that

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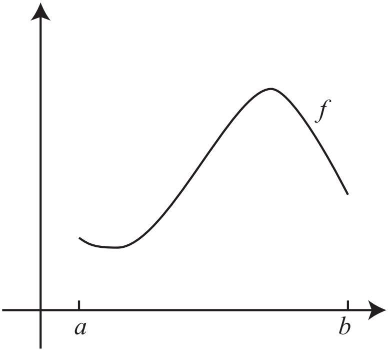

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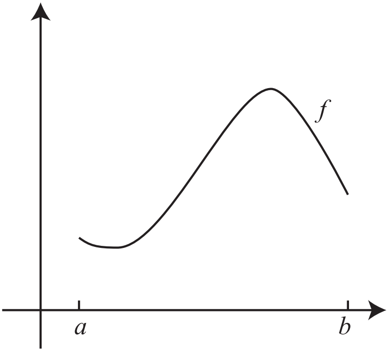

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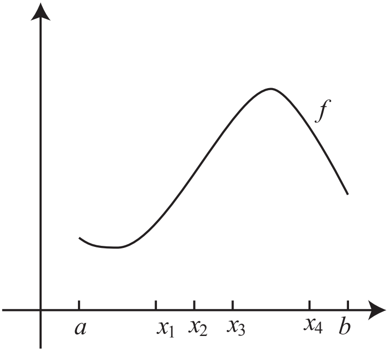
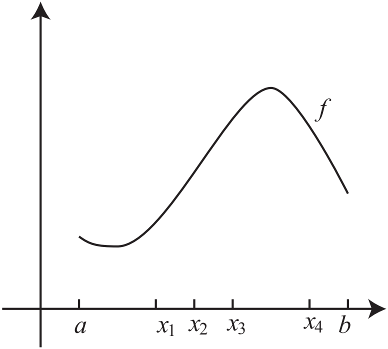
Write $\mathcal{P} = \mathcal{P}^{(i)}$.



Proof (continued)



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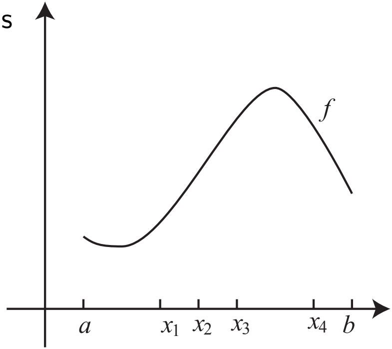

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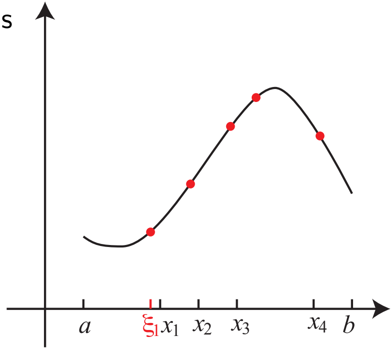

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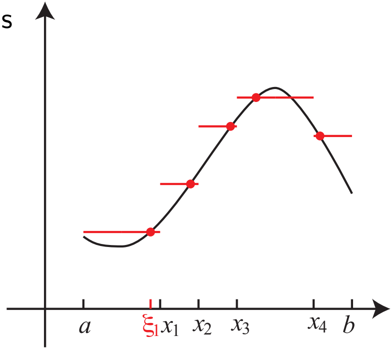

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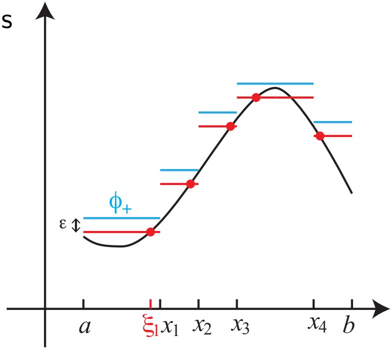

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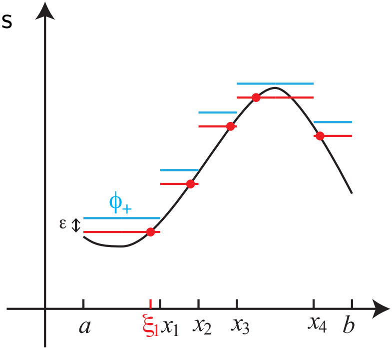

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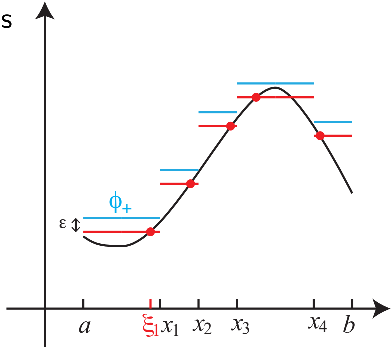

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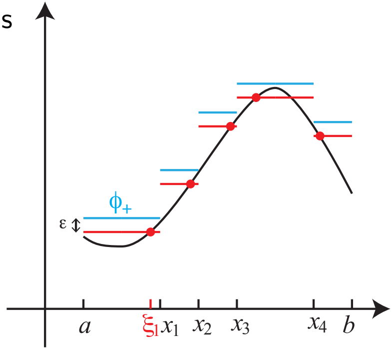

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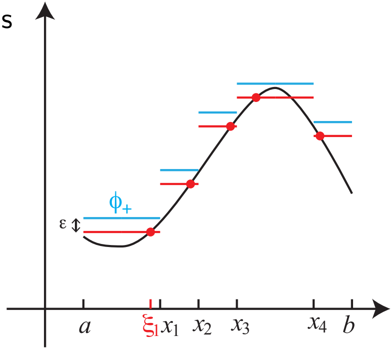

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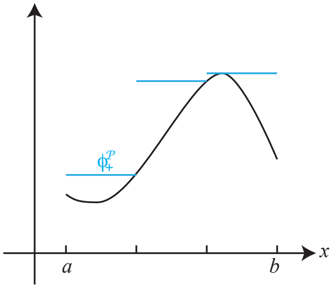

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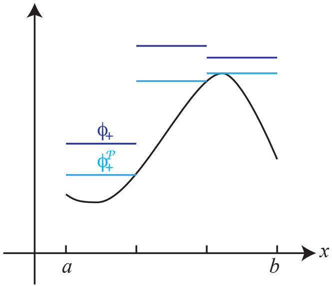

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Therefore, f is integrable if and only if for every $\varepsilon > 0$, there is a partition $\mathcal{P}$ such $I(\phi_+^{\mathcal{P}}) - I(\phi_-^{\mathcal{P}}) < \varepsilon$.

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Set $\delta := \varepsilon / nM$ where $|f(x)| \le M$ for all $x \in [a, b]$.

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Set $\delta := \varepsilon / nM$ where $|f(x)| \leq M$ for all $x \in [a, b]$.
Let $\mathcal{P}' : a = x'_0 \leq x'_1 \leq \cdots \leq x'_{n'} = b$ be any partition with mesh$(\mathcal{P}') \leq \delta$.
We will show that for any Riemann sum $\Sigma(f, \mathcal{P}', \vec{\xi'})$,

$$\int_a^b f - 5\varepsilon \leq \Sigma(f, \mathcal{P}', \vec{\xi'}) \leq \int_a^b f + 5\varepsilon.$$

This will conclude the proof.

Proof of Proposition 3.2 (continued)

$$\Sigma(f, \mathcal{P}', \vec{\xi'})$$

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$$\Sigma(f, \mathcal{P}', \vec{\xi'}) = \sum_{j=1}^{n'} f(\xi'_j)(x'_j - x'_{j-1})$$

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where the step function ψ is defined to be $f(\xi_j')$ on (x_{j-1}', x_j') and $f(x_j')$ at the x_j'.

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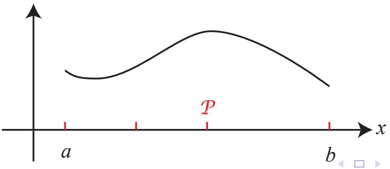
Let us compare ψ and the optimal majorant $\phi_+^{\mathcal{P}}$.



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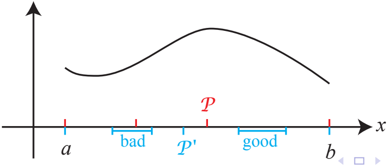

Proof of Proposition 3.2 (continued)

$$\Sigma(f, \mathcal{P}', \vec{\xi}') = \sum_{j=1}^{n'} f(\xi'_j)(x'_j - x'_{j-1}) = I(\psi),$$

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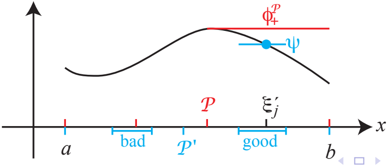
If j is good then, for $t \in (x'_{j-1}, x'_j)$,

$$\psi(t) = f(\xi'_j) \leq \sup_{x \in [x'_{j-1}, x'_j]} f(x) \leq \sup_{x \in (x_{i-1}, x_i)} f(x) = \phi_+^{\mathcal{P}}(t).$$



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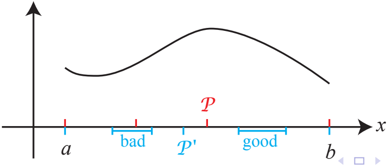

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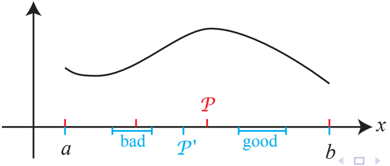

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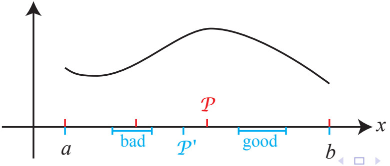

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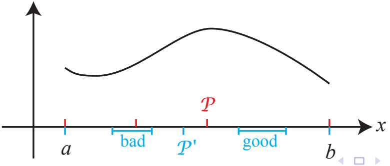

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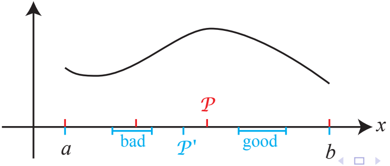

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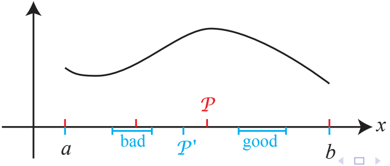
Considering both the good and bad intervals,

$$\Sigma(f, \mathcal{P}', \vec{\xi}') = I(\psi) \leq I(\phi_+^{\mathcal{P}}) + 2M \cdot \frac{2\varepsilon}{M} = I(\phi_+^{\mathcal{P}}) + 4\varepsilon$$



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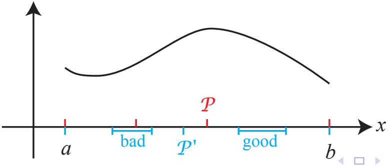

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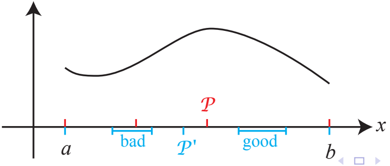

Chapter 4A: The fundamental theorem of calculus

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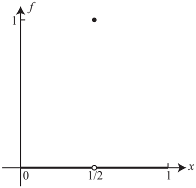
An example

Let $f : [0,1] \to \mathbb{R}$ be

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$



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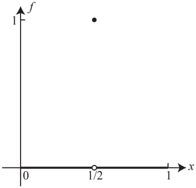

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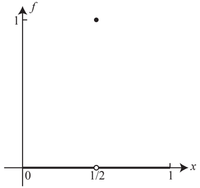

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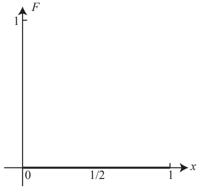

Then F is identically zero.

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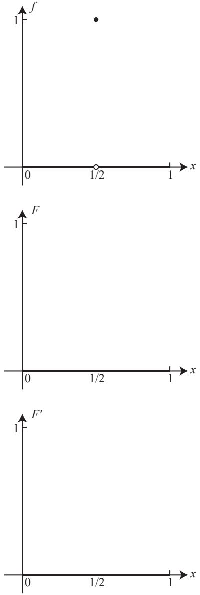

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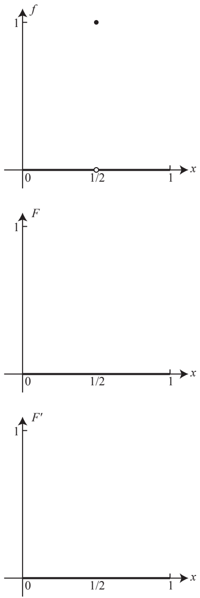

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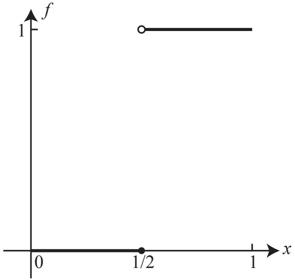

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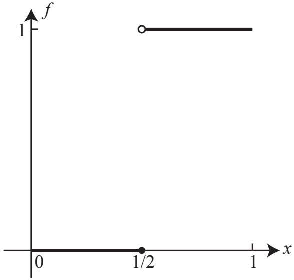

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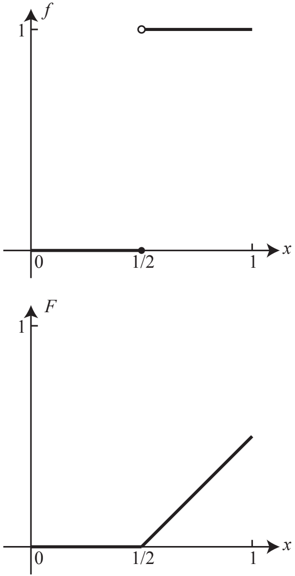

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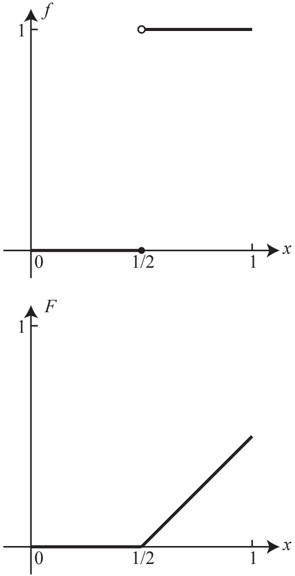
So, F is not differentable at $x = \frac{1}{2}$.



The first fundamental theorem of calculus

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Theorem 4.1. Suppose that f is integrable on (a, b). Define a new function $F : [a, b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f.$$

Then F is continuous.

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Hence, F is Lipschitz, hence uniformly continuous, hence continuous.

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Essentially the same argument works for $h < 0$. Hence, F is differentiable at c with derivative $f(c)$.

Chapter 4B: The second fundamental theorem of calculus

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Example. Let $F : [-1, 1] \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

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Then F is differentable everywhere, with $f = F'$ given by

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In particular, f is unbounded on any interval containing 0, and so it has no majorants and is not integrable according to our definition.

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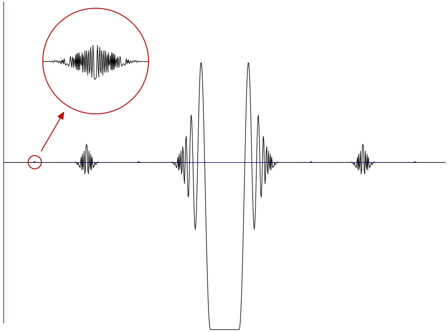

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Proof. Let $\mathcal{P}$ be a partition, $a = x_0 < x_1 < \cdots < x_n = b$.

The second fundamental theorem of calculus, applications

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Summing from $i = 1$ to n gives

$$\Sigma(F'; \mathcal{P}, \xi) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(b) - F(a).$$

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By Proposition 1.19 and the assumption that f', g' are integrable, F' is integrable on (a, b).
Applying the fundamental theorem gives

$$\int_a^b F' = F(b) - F(a).$$

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Written out in full:

$$\int_a^b f(x)dx = \int_c^d f(\phi(t))\frac{d\phi}{dt}dt.$$

Substitution (proof).

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Since f is continuous on $[a, b]$, it is integrable. The first fundamental theorem of calculus implies that its antiderivative

$$F(x) := \int_a^x f$$

is continuous on $[a, b]$, differentiable on (a, b) and that $F' = f$.

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By the chain rule and the fact that $\phi((c,d)) \subset (a,b)$, $F \circ \phi$ is differentiable on (c,d), and

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By the second form of the fundamental theorem,

$$\begin{aligned}\int_c^d (f \circ \phi)\phi' &= \int_c^d (F \circ \phi)' \\ &= (F \circ \phi)(d) - (F \circ \phi)(c) \\ &= F(b) - F(a) \\ &= F(b) = \int_a^b f.\end{aligned}$$

Chapter 5A: Interchanging limits and integration

Interchanging limits and integration

$$\lim_{n\to\infty} \int_a^b f_n \stackrel{?}{=} \int_a^b \lim_{n\to\infty} f_n$$

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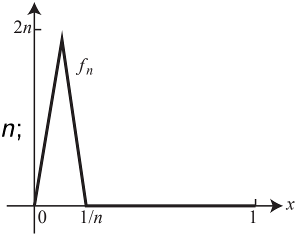
Let $f_n : [0,1] \to \mathbb{R}$ be

$$f_n(x) = \begin{cases} 2n^2 x & \text{if } x \leq 1/(2n); \\ 2n - 2n^2 x & \text{if } 1/(2n) < x < 1/n; \\ 0 & \text{otherwise} \end{cases}$$



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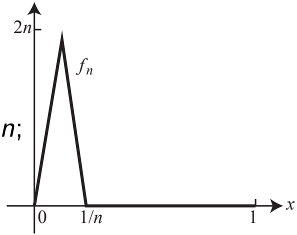

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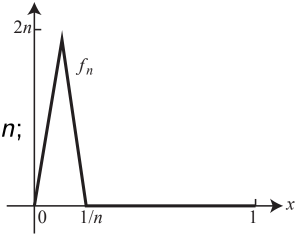

Then f_n converges pointwise to the zero function.
But $\int_0^1 f_n = 1$.

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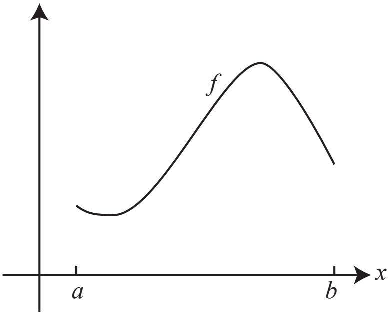

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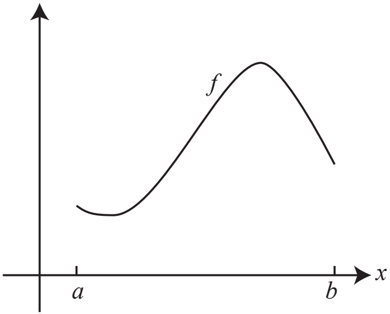

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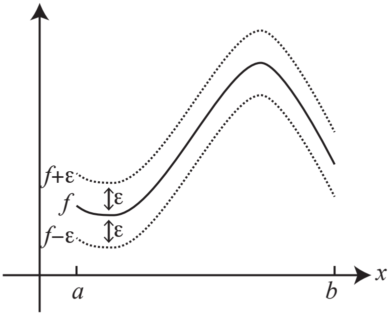

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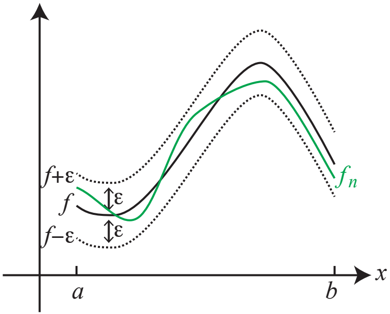

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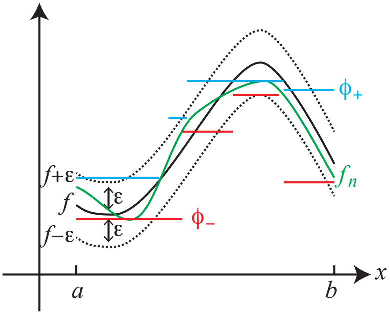
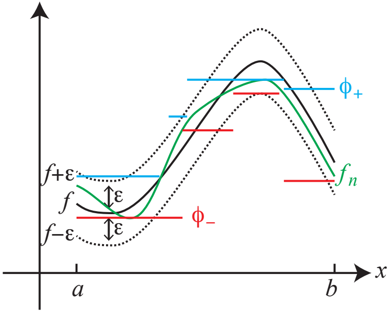
Since $f_n \to f$ uniformly, there is some choice of n such that we have $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in [a, b]$.

Now f_n is integrable, and so there is a majorant ϕ_+ and a minorant ϕ_- for f_n with $I(\phi_+) - I(\phi_-) \leq \varepsilon$.



Proof (continued)



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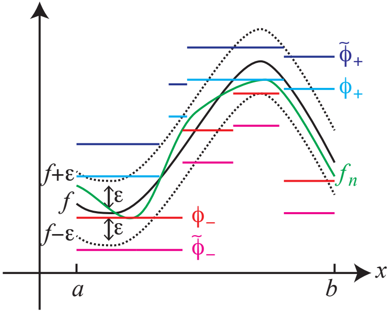

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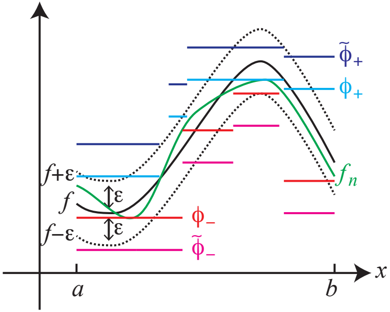

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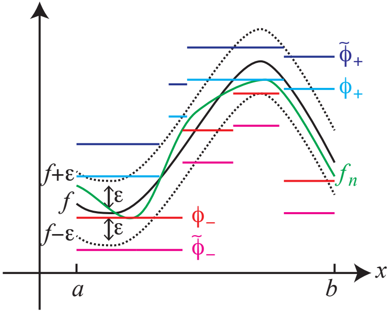

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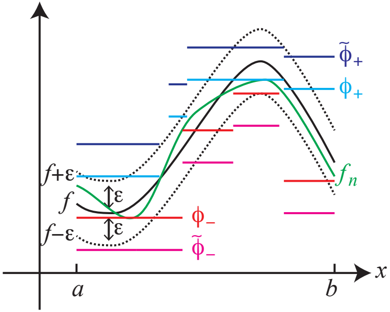

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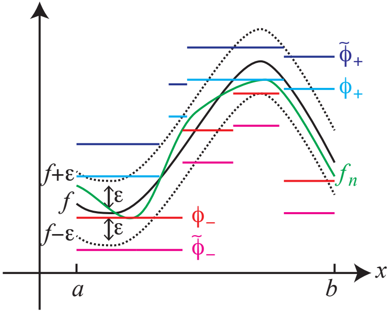

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$$\left|\int_a^b f_n - \int_a^b f\right| \leq \int_a^b |f_n - f| \leq (b-a) \sup_{x\in[a,b]} |f_n(x) - f(x)|.$$

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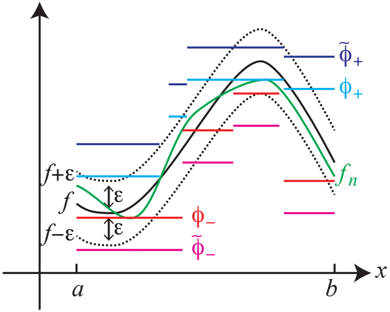

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Since $f_n \to f$ uniformly, it follows that

$$\lim_{n\to\infty} \left| \int_a^b f_n - \int_a^b f \right| = 0.$$

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Integration and sums

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Proof. This is immediate from the Weierstrass M-test and Theorem 5.2, applied with $f_n = \sum_{i=1}^{n} \phi_i$. □

Chapter 5B: Interchanging limits and differentiation

An example

Take $f_n(x) = \frac{1}{n} \sin(n^2 x)$.

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So, $f_n'(\pi/4)$ does not converge as $n \to \infty$.

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Then f is differentiable and $f' = g$. In particular, $\lim_{n\to\infty} f_n' = (\lim_{n\to\infty} f_n)'$.

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Since $f_n' \to g$ uniformly, $\lim_{n\to\infty} \int_a^x f_n'(t)dt = \int_a^x g(t)dt$ by Theorem 5.1.

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Since $f_n \to f$ pointwise, $\lim_{n\to\infty} \int_a^x f_n'(t)dt = f(x) - f(a)$.
Since $f_n' \to g$ uniformly, $\lim_{n\to\infty} \int_a^x f_n'(t)dt = \int_a^x g(t)dt$ by Theorem 5.1. Thus

$$F(x) = \int_a^x g(t)dt = f(x) - f(a).$$

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$$F(x) = \int_a^x g(t)dt = f(x) - f(a).$$

It follows immediately that f is differentiable and that its derivative is the same as that of F, namely g. □

Term-by-term differentation of series

Corollary 5.6. Suppose we have a sequence of continuous functions $\phi_i : [a, b] \to \mathbb{R}$, continuously differentiable on (a, b), with $\sum_i \phi_i$ converging pointwise. Suppose that $|\phi_i'(x)| \leq M_i$ for all $x \in (a, b)$, where $\sum_i M_i < \infty$. Then $\sum \phi_i$ is differentiable and

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$$(\sum_i \phi_i)' = \sum_i \phi_i'.$$

Proof. Apply Proposition 5.5 with $f_n := \sum_{i=1}^n \phi_i$. By the Weierstrass M-test, $f_n' = \sum_{i=1}^n \phi_i'$ converges uniformly to some bounded function, which we may call g. $\square$

Chapter 5C: Radius of convergence

Power series and radius of convergence

Definition. Now suppose we have a sequence $(a_i)_{i=0}^{\infty}$ of real numbers. Then the expression $\sum_{i=0}^{\infty} a_i x^i$ is called a (formal) power series.

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Definition. Given a formal power series $\sum_i a_i x^i$, we define its radius of convergence R to be the supremum of all $|x|$ for which the sum $\sum_{i=0}^{\infty} |a_i x^i|$ converges. If this sum converges for all x, we write $R = \infty$.

Main theorem

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Geometric series

Lemma. Suppose that $0 \le \lambda < 1$. Then $\sum_{i=0}^{\infty} \lambda^i$ and $\sum_{i=1}^{\infty} i\lambda^{i-1}$ both converge.

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For the second statement, we differentiate the geometric series formula. This gives

$$\sum_{i=1}^{n-1} i\lambda^{i-1} = \frac{1+(n-1)\lambda^n - n\lambda^{n-1}}{(1-\lambda)^2},$$

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which tends to $\frac{1}{(1-\lambda)^2}$ as $n \to \infty$. $\square$

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Conditions of Corollary 5.6:

1. ϕ_i continuous of $[a, b]$ and continuously differentiable on (a, b);
2. $\sum_i \phi_i$ converging pointwise;
3. $|\phi_i'(x)| \leq M_i$ for all $x \in (a, b)$, where $\sum_i M_i < \infty$.

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(1) is immediate.

(2) Let R_0 satisfy $R_1 < R_0 < R$. By assumption, $\sum_i |a_i R_0^i|$ converges, and so $|a_i R_0^i| \leq K$ uniformly in i. Then if $x \in [a, b]$ we have

$$|\phi_i(x)| \leq K(\frac{R_1}{R_0})^i$$

and so by the geometric series lemma (first part), $\sum_i \phi_i(x)$ converges pointwise.

Proof (continued)

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(3) If $x \in [a, b]$, then

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It now follows from Corollary 5.6 that f is differentiable on $(-R_1, R_1)$, and that is derivative is given by term-by-term differentiation of the power series for f.

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By the geometric series lemma, the radius of convergence of the power series for f' is at least R_1.

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By the geometric series lemma, the radius of convergence of the power series for f' is at least R_1. Since $R_1 < R$ was arbitrary, the radius of convergence of this power series is at least R. $\square$

Chapter 6A: The exponential function

A simple differential equation

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Proof. Since f is continuous, it attains its maximum value on $[0, \frac{1}{2}]$ at some point x.

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We may now apply the same argument to $g(x) = f(x - \frac{1}{2})$, which satisfies $g' = g$ and $g(0) = 0$. We conclude that g is identically zero on $[0, \frac{1}{2}]$, and hence that f is identically zero on $[\frac{1}{2}, 1]$ and hence on $[0, 1]$.

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We may now apply the same argument to $g(x) = f(x - \frac{1}{2})$, which satisfies $g' = g$ and $g(0) = 0$. We conclude that g is identically zero on $[0, \frac{1}{2}]$, and hence that f is identically zero on $[\frac{1}{2}, 1]$ and hence on $[0, 1]$. Continuing in this manner eventually shows that f is identically zero on the whole of $\mathbb{R}$. $\square$

Simple properties of the exponential function

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Simple properties of the exponential function

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2. We have $e(x) > 0$ for all $x \in \mathbb{R}$.
3. We have $e(x + y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.

Proof of 1

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Term-by-term differentiation gives the same series back again. So by Theorem 5.9, it is enough to show that the radius of convergence is infinite ie that $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for all x. This is a simple consequence of the ratio test (limit form):

$$\frac{x^{k+1}}{(k+1)!} \Big/ \frac{x^k}{k!} = \frac{x}{k+1} \to 0 \quad \text{as } k \to \infty.$$

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By Lemma 6.1, f is identically zero and hence so is e. But this is a contradiction, as e is clearly not identically zero (for example $e(0) = 1$).

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By Lemma 6.1, f is identically zero and hence so is e. But this is a contradiction, as e is clearly not identically zero (for example $e(0) = 1$).
Thus e never vanishes. Since it is continuous, and positive somewhere, the intermediate value theorem implies that it is positive everywhere.

Proof of 3

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Therefore the function $f := e - \tilde{e}$ satisfies the hypotheses of Lemma 6.1.

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Therefore the function $f := e - \tilde{e}$ satisfies the hypotheses of Lemma 6.1. It follows that $\tilde{e}(x) = e(x)$. □

Chapter 6B: The logarithm function

The logarithm function

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1. L is differentiable with derivative $\frac{1}{x}$ at each $x > 0$;
2. $L(e^t) = t$ for all $t \in \mathbb{R}$.

(When $x < 1$, we define $\int_b^a f$ to be $-\int_a^b f$ when $a < b$.)

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Let $c > 0$ and write

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It is easy to check that this holds for any $c > 0$.
Then we may apply the fundamental theorem of calculus to get that $L'(x) = \frac{1}{x}$ for any $x > c$. Since c was arbitrary, the result follows.

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Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $\phi : [c, d] \to [a, b]$ is continuous on $[c, d]$, has $\phi(c) = a$ and $\phi(d) = b$, and maps (c, d) to (a, b). Suppose moreover that ϕ is differentiable on (c, d) and that its derivative ϕ' is integrable on this interval. Then

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

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Set $f(y) = \frac{1}{y}$ and $\phi(t) = e^t$.
Note that $f(\phi(t))\phi'(t) = 1$, since $\phi' = \phi$. We therefore have

$$\int_1^{e^x} \frac{dt}{t} = \int_0^x (f \circ \phi)\phi' = x.$$

Chapter 7: Improper integrals

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$$\int_{\varepsilon}^{1} \log x \; dx = [x \log x - x]_{\varepsilon}^{1} = -1 - \varepsilon \log \varepsilon - \varepsilon.$$

We claim that $\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = 0$.

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We claim that $\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = 0$. Once this is shown, it follows that

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \log x \; dx = -1.$$

This will often be written as

$$\int_{0}^{1} \log x \; dx = -1,$$

but strictly speaking, as remarked above, this is not an integral as discussed in this course.

Proof of claim

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$$|\int_\varepsilon^{\sqrt{\varepsilon}} \frac{dx}{x}| \leq \frac{1}{\sqrt{\varepsilon}}.$$

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It follows that

$$|\log \varepsilon| \leq \frac{2}{\sqrt{\varepsilon}},$$

from which the claim follows immediately. □

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This is invariably written

$$\int_1^\infty \frac{1}{x^2}\,dx = 1.$$

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By this 'double limit', we formally mean the following: For all $\varepsilon' > 0$, there are $N \in (0, \infty)$ and $\delta > 0$ such that for all $K > N$ and all $\varepsilon \in (0, \delta)$,

$$\left| \int_\varepsilon^K f(x)dx - 0 \right| < \varepsilon'.$$

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$$I_{\varepsilon,\varepsilon'} := \int_{\varepsilon}^{1} f(x)dx + \int_{-1}^{-\varepsilon'} f(x)dx = \log \frac{\varepsilon'}{\varepsilon}.$$

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The *Cauchy principal value* (PV) is the limit $\lim_{\varepsilon \to 0} I_{\varepsilon,\varepsilon} = 0$.

It is *not* appropriate to write $\int_{-1}^{1} \frac{1}{x} dx = 0$; one could possibly write PV $\int_{-1}^{1} \frac{1}{x} dx = 0$.

Example 7.5.

Similarly to the last example, one should not write $\int_{-\infty}^{\infty} \sin x \; dx = 0$, even though $\lim_{K\to\infty} \int_{-K}^{K} \sin x \; dx = 0$ (because sin is an odd function). In this case, $\lim_{K,K'\to\infty} \int_{-K'}^{K} \sin x \; dx$ does not exist.

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One could maybe write

$$\mathrm{PV} \int_{-\infty}^{\infty} \sin x \; dx = 0,$$

but I would not be tempted to do so.