C3.10 Additive and Combinatorial NT
Lecture 1: Introduction and preliminaries

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About this course

- The course is divided into two (mostly disjoint) parts: Additive Number Theory and Additive Combinatorics
- Part A Number Theory assumed, attending C3.8 Analytic NT useful but not necessary. B4.4 Fourier transform may also be useful, but not assumed.
- In lectures, we will complement lecture notes, not necessarily replicate all proofs.
- It may be a good idea to pause the video when an important proof or idea is being presented!
- The slides will be made available.
- Questions about exercises can be sent to your tutor, corrections to lecture notes are welcome to `joni.teravainen @ maths.ox.ac.uk`

Highlights of the course

Theorem (Lagrange)

Every positive integer can be written as the sum of four squares of integers.

Theorem (Waring's problem; Hilbert & Hardy–Littlewood)

For every $k \geq 2$, there exists s such that every integer is the sum of s kth powers of nonnegative integers.

Highlights of the course

Theorem (Roth)

Let $A \subset \mathbb{N}$ have positive upper density, that is, $\limsup_{N\to\infty} \frac{|A\cap[1,N]|}{N} > 0$. Then A contains infinitely many nontrivial three-term arithmetic progressions.

Theorem (Freiman)

Let $A \subset \mathbb{Z}$ be a set of small doubling, that is, $|A + A| \leq K|A|$ for some $K \geq 1$. Then A is contained in a *generalized arithmetic progression* P such that the dimension of P and density $|P|/|A|$ are both bounded in terms of K.

Landau and Vinogradov notation

The following notation will be used extensively:

$A = O(B)$ if there is a constant $C > 0$ such that $|A| \leq C|B|$.

Usually, $A = A(x)$, $B = B(x)$, and we are interested in $x \to \infty$.

Example

$P(x) = O(Q(x))$ whenever $\deg P < \deg Q$, $\sin x = O(1)$ $(x \to \infty)$

We also denote $A = O(B)$ by $A \ll B$ (or $B \gg A$).

Example

$x^{1-\varepsilon} \ll_\varepsilon \frac{x}{\log x} \ll x \ll x/100 - 100 \ll x \log x \ll x^2$ $(x \to \infty)$.

We denote $A = o(B)$ if $B(x)/A(x) \to 0$ as $x \to \infty$.

Example

$1/\log x = o(1)$, $\log x = x^{o(1)} (x \to \infty)$.

See Sheet 0 for more practice on using these symbols.

Preliminaries: Fourier transform

We will need the Fourier transform on three groups $(\mathbb{Z}/q\mathbb{Z}, \mathbb{Z}, \mathbb{R})$.

Basic knowledge of what these are and how they work is essential for us, but we don't need their finer properties.

Section 1 of the lecture notes contains all you need about them.

Fourier transforms

If $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$, define

$$\hat{f}(r) = \sum_{x \in \mathbb{Z}/q\mathbb{Z}} f(n) e(-rx/q), \quad e(x) := e^{2\pi i x}.$$

If $f : \mathbb{Z} \to \mathbb{C}$ is "nice", define

$$\hat{f}(\theta) = \sum_{n \in \mathbb{Z}} f(n) e(-n\theta).$$

If $f : \mathbb{R} \to \mathbb{C}$ is "nice", define

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e(-\xi x)\, dx.$$

Preliminaries Fourier transform

Perhaps the most important property for us is:

Parseval's formula

If $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$, we have

$$\sum_{r \in \mathbb{Z}/q\mathbb{Z}} |\hat{f}(r)|^2 = \frac{1}{q} \sum_{x \in \mathbb{Z}/q\mathbb{Z}} |f(n)|^2$$

If $f : \mathbb{Z} \to \mathbb{C}$ is "nice", we have

$$\int_0^1 |\hat{f}(\theta)|^2 \, d\theta = \sum_{n \in \mathbb{Z}} |f(n)|^2.$$

If $f : \mathbb{R} \to \mathbb{C}$ is "nice", we have

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, d\xi = \int_{\mathbb{R}} |f(x)|^2 \, dx.$$

The proofs are straightforward.

See Sheet 0 for some practice problems.