C3.10 Additive and Combinatorial NT
Lecture 3: Waring's problem and the circle method

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Waring's problem

Lagrange's theorem: any $n \geq 1$ is the sum of four squares.

Waring (1770) conjectured:

Waring's problem

Let $k \geq 2$. Then there exists $s = s(k)$ such that every positive integer is the sum of s kth powers of nonngeative integers.

This was finally proved by Hilbert (1909) and Hardy and Littlewood (1920s).
It suffices to prove that every *large* n is a sum of kth powers.

Definition

Denote by $G(k)$ be the least integer s such that every large enough integer is the sum of s kth powers of nonnegative integers.

Theorem (Solution to Waring's problem)

$G(k)$ is finite, and moreover $G(k) \leq 100^k$.

Waring's problem

In Section 8 of lecture notes (Hua's lemma), it is proved that $G(k) \leq 2^k + 1$.

We have the easy lower bound $G(k) \geq k + 1$ (Sheet 1), but the best upper bound is a deep result of Wooley: $G(k) \leq k \log k + k \log \log k + O(k)$.

$G(2) = 4$, $G(4) = 16$, but all other values unknown! **Open problem:** Is $G(3) = 4$?.

We will follow (a modern version of) Hardy and Littlewood's proof. This is based on their influential *circle method*.

Waring's problem: Asymptotic formula

We will in fact prove an asymptotic formula:

Theorem (Asymptotics for Waring's problem)

Let $r_{k,s}(N)$ be the number of representations of N as $x_1^k + \ldots + x_s^k$, $x_i \geq 0$. Suppose that $s \geq 100^k$. Then

$$r_{k,s}(N) = \mathfrak{S}_{k,s}(N) N^{s/k-1} + o(N^{s/k-1}),$$

where the *singular series* is

$$\mathfrak{S}_{k,s}(N) = \beta_\infty \prod_p \beta_p(N),$$

and $\beta_p(N)$ is the local density of solutions,

$$\beta_p(N) = \lim_{n \to \infty} p^{-n(s-1)} |\{(x_1, \ldots, x_s) \in (\mathbb{Z}/p^n\mathbb{Z})^s : \sum_{i \leq s} x_i^k \equiv N\,(p^n)\}|,$$

and the Archimedean density is

$$\beta_\infty = \Gamma(1 + 1/k)^s / \Gamma(s/k).$$

Waring's problem: Asymptotic formula

The asymptotic formula is complemented by

Theorem (Singular series)

For $s \geq k^4$ we have $1 \ll \mathfrak{S}_{k,s}(N) \ll 1$ (i.e., $\mathfrak{S}_{k,s}(N) \asymp 1$).

Interpretation of the asymptotic formula:

It implies that $r_{k,s}(N) \asymp N^{s/k-1}$. This is the expected order of magnitude, since we can show by elementary means (Sheet 1) that

$$c_{s,k} N^{s/k} \leq \sum_{n \leq N} r_{k,s}(N) \leq C_{s,k} N^{s/k}.$$

The asymptotic formula is a *local-to-global principle*: If $P(x_1, \dots, x_s) = x_1^k + \dots + x_s^k$, then

$$\begin{aligned} &|\{\bar{x} \in \mathbb{N}_0^s : P(\bar{x}) = N\}| \\ &\sim \beta_\infty N^{s/k-1} \prod_p \lim_{n \to \infty} \Pr(\bar{x} \in (\mathbb{Z}/p^n\mathbb{Z})^s : P(\bar{x}) \equiv N \pmod{p^n}), \end{aligned}$$

and one can show that $\beta_\infty N^{s/k-1} = \mathrm{Area}(\bar{x} \in \mathbb{R}_{\geq 0} : P(\bar{x}) = N)$.

The Hardy–Littlewood circle method

This method will underlay what we do in the next few lectures.

The details are somewhat complicated, but the ideas are important.

Very roughly, the circle method tells us that

Counting sol's to $a_1 + \ldots + a_s = N, a_i \in A \leftrightarrow$ Estimating $\widehat{1_A}(\theta)\ \forall\theta \in \mathbb{T}$.

However, the number of variables here needs to be large enough (depending on the problem at hand) for this approach to work.

Thus, the key object will be the the Fourier transform (or exponential sum)

$$\widehat{1_A}(\theta) = \sum_{n \in A} e(-\theta n).$$

The Hardy–Littlewood circle method

More precisely, we have the following

Theorem

If $A \subset \mathbb{Z}$ is finite, then

$$|\{(a_1, ..., a_s) \in A^s : \ a_1 + ... + a_s = N\}| = \int_0^1 \widehat{1_A}(\theta)^s e(N\theta)\, d\theta.$$

Proof: The LHS is

$$\sum_{a_1,..,a_s \in A} 1_{a_1+...+a_s-N=0}.$$

Use $1_{n=0} = \int_0^1 e(nx)\, dx$ and change order of integral and sum. □

Corollary

Denoting $X = \{n^k : \ n \leq N^{1/k}\}$,

$$r_{k,s}(N) = \int_0^1 \widehat{1_X}(\theta)^s e(N\theta)\, d\theta.$$

The Hardy–Littlewood circle method

We now need to estimate $\widehat{1_X}(\theta)$ uniformly in θ.

Heuristic: $\widehat{1_X}(\theta)$ is typically large when $\theta \approx a/q$ with a, q "small".

If θ is "far from" such rationals ("highly irrational"), then $\widehat{1_X}(\theta)$ is small.

Example

Suppose $\theta = 1/3$, $k = 2$. Then

$$\widehat{1_X}(\theta) = \sum_{n \leq N^{1/2}} e(n^2/3) = (1/3 + 2e(1/3)/3 + o(1))N^{1/2} \gg N^{1/2}.$$

Similarly, if $\theta = a/q$ with $q = O(1)$, then

$$\widehat{1_X}(\theta) = \sum_{n \leq N^{1/2}} e(an^2/q) \gg N^{1/2}.$$

Lastly, if we perturb θ by c/N for small $c > 0$, nothing changes.

The Hardy–Littlewood circle method

Example

Let $\theta = \sqrt{2}, k = 2$. Then

$$\widehat{1_X}(\theta) = \sum_{n \leq N^{1/2}} e(\sqrt{2}n^2).$$

We expect the sequence $\sqrt{2}n^2 \pmod 1$ to be equidistributed in $[0, 1]$, and therefore we expect $\widehat{1_X}(\theta) = o(N^{1/2})$. Again, the same holds if we perturb θ by $o(1/N)$.

The Hardy–Littlewood circle method

Let $\|\theta\|$ be the distance from θ to the nearest integer. We identity $\mathbb{T}$ and $[0, 1)$.

To talk rigorously about rationals of small denominator, we define:

Definition

Set $\eta := 1/(10k)$. Define the *major arcs*

$$\mathfrak{M} = \bigcup_{\substack{q \le N^\eta \\ a \in (\mathbb{Z}/q\mathbb{Z})^*}} \mathfrak{M}_{a,q},$$

where

$$\mathfrak{M}_{a,q} := \{\theta \in \mathbb{T} : |\theta - \frac{a}{q}| \le N^{-1+2\eta}\}.$$

Define the *minor arcs* $\mathfrak{m}$ to be $\mathbb{T} \setminus \mathfrak{M}$.

Roughly, $\theta \in \mathfrak{M}$ if there is $q \lesssim 1$ such that $\|q\theta\| \lesssim 1/N$.

The Hardy–Littlewood circle method

The idea of the circle method is to evaluate

$$\int_{\mathfrak{M}} \widehat{1_X}(\theta)^s e(N\theta)\,d\theta \quad \text{and} \int_{\mathfrak{m}} \widehat{1_X}(\theta)^s e(N\theta)\,d\theta.$$

Why is this called the circle method? Think of $\mathbb{T}$ as the unit circle of $\mathbb{C}$ (via $x \mapsto e(x)$), and draw a small arc around every point of the unit circle having angle a rational multiple of 2π.

Prop. 3.2.1 (Major arcs)

Let $s \geq 2k+1$. Then,

$$\int_{\mathfrak{M}} \widehat{1_X}(\theta)^s e(N\theta)\,d\theta = \mathfrak{S}_{k,s}(N) N^{s/k-1} + o(N^{s/k-1}).$$

Prop. 3.2.2 (Minor arcs)

Let $s \geq 100^k$. Then,

$$\int_{\mathfrak{m}} \widehat{1_X}(\theta)^s e(N\theta)\,d\theta = o(N^{s/k-1}).$$

These propositions, together with bounds for the singular series, will lead to the solution of Waring's problem.