C3.10 Additive and Combinatorial NT
Lecture 4: Waring's problem: the minor arcs

Joni Teräväinen

Mathematical Institute

Circle method

Recall we need to prove three things to solve Waring's problem:

Prop. 3.2.1 (Major arcs)

Let $s \geq 2k+1$, $X = \{n^k : \ n \leq N^{1/k}\}$. Then,

$$\int_{\mathfrak{M}} \widehat{1_X}(\theta)^s e(N\theta)\, d\theta = \mathfrak{S}_{k,s}(N) N^{s/k-1} + o(N^{s/k-1}).$$

Prop. 3.2.2 (Minor arcs)

Let $s \geq 100^k$. Then,

$$\int_{\mathfrak{m}} \widehat{1_X}(\theta)^s e(N\theta)\, d\theta = o(N^{s/k-1}).$$

Prop. 3.1.1 (Singular series)

Let $s \geq k^4$. Then $1 \ll \mathfrak{S}_{k,s}(N) \ll 1$ (i.e., $\mathfrak{S}_{k,s}(N) \asymp 1$).

The minor arcs

By the triangle inequality,

$$\left| \int_{\mathfrak{m}} \widehat{1_X}(\theta)^s e(N\theta)\, d\theta \right| \le \sup_{\theta \in \mathfrak{m}} |\widehat{1_X}(\theta)|^s,$$

so the minor arc proposition will follow from

Prop. 4.0.1 (Pointwise estimate)

Let $\varepsilon = 100^{-k}$. Then

$$\sup_{\theta \in \mathfrak{m}} |\widehat{1_X}(\theta)| \ll N^{1/k - \varepsilon}.$$

We will deduce this from a slightly more general bound for *exponential sums* $\sum_{x \in I} e(P(x))$ (Weyl sums).

Weyl sums

Theorem 4.2.1 (estimate for Weyl sums)

Set $C_k := 10^k$. Let δ be sufficiently small in terms of k, and suppose that $L > \delta^{-C_k}$. Let $I \subseteq \mathbb{Z}$ be an interval of length at most L. Let $P : \mathbb{Z} \to \mathbb{R}$, $P(x) = \alpha x^k + \cdots$ be a polynomial of degree k. Suppose that $|\sum_{x \in I} e(P(x))| \geq \delta L$. Then there is $q \leq \delta^{-C_k}$ such that $\|q\alpha\| \leq \delta^{-C_k} L^{-k}$.

Deduction of Prop. 4.0.1: Take $I = \{n \leq N^{1/k}\}$, $L = \lfloor N^{1/k} \rfloor$, $\delta = N^{-\varepsilon}$ ($\varepsilon = 100^{-k}$). Then if $\theta \in \mathbb{R}$ satisfies $|\widehat{1_X}(\theta)| > \delta N^{1/k}$, there exists $q \leq \delta^{-C_k} \leq N^{\eta}$ ($\eta = 1/(10k)$) such that $\|q\theta\| \leq \delta^{-C_k} L^{-k} \ll N^{\eta - 1}$, so $\theta \in \mathfrak{M}$.

Vinogradov's lemma

The proof of Theorem 4.2.1 (Weyl sums) makes use of a lemma on the distribution of $n\alpha \pmod 1$.

If α is "highly irrational", then we expect uniform distribution: $\|\alpha n\| \leq \delta$ for proportion 2δ for integers $n \leq N$. The next lemma is a converse to this: if $\|\alpha n\|$ is far from uniformly distributed, then α is "highly rational".

Lemma (Vinogradov)

There is an absolute constant C with the following property. Suppose $\alpha \in \mathbb{R}$ and that $I \subset \mathbb{Z}$ is an interval with $|I| = N$. Suppose δ_1, δ_2 are positive quantities with $\delta_2 > C\delta_1$, and suppose that there are at least $\delta_2 N$ elements $n \in I$ for which $\|\alpha n\| \leq \delta_1$. Suppose $N \geq C/\delta_2$. Then there is $1 \leq q \leq C/\delta_2$ such that $\|\alpha q\| \leq C\delta_1/\delta_2 N$.

Roughly: If $\|\alpha n\| \leq \delta$ for $> 1000\delta N$ integers and $N \gg_\delta 1$, there is $q \ll_\delta 1$ s.t. $\|q\alpha\| \ll_\delta 1/N$.

Proof of Vinogradov's lemma

We start with a well-known lemma.

Theorem (Dirichlet)

Let $\alpha \in \mathbb{R}$ and $Q \geq 1$. Then there exists $1 \leq q \leq Q$ such that $\|q\alpha\| \leq 1/Q$.

Proof

Apply the pigeonhole principle to $\alpha, 2\alpha, ..., Q\alpha \pmod 1$.

Proof of Vinogradov's lemma

The proof of Vinogradov's lemma is in steps. Let
$S = \{n \in I : \|\alpha n\| \leq \delta_1\}$.

Step 1: Reduction to the case $I = [1, N] \cap \mathbb{Z}$.

This is just a change of variables.

Step 2: Applying Dirichlet's theorem.

Apply Dirichlet's thm with $Q = 4N$. Thus, $\exists 1 \leq q \leq 4N$ such that $\|\alpha q\| \leq 1/(4N)$. Hence, $\exists a$ coprime to q such that $|\alpha - a/q| \leq 1/(4qN)$. This gives

$$\|\alpha n\| \leq \|an/q\| + 1/(4q), \quad \text{for} \quad n \in S.$$

Step 3: Reducing q. The number of solutions n to $\|an/q\| \leq \delta_1 + 1/(4q)$ is

$$\begin{aligned} &\leq (N/q + 1)|\{1 \leq n \leq q : \ \|an/q\| \leq \delta_1 + 1/(4q)\}| \\ &\leq (N/q + 1)(2q(\delta_1 + 1/(4q)) + 1). \end{aligned}$$

This should be $\geq \delta_2 N$, so with a bit of algebra $q \leq 16/\delta_2$.

Proof of Vinogradov's lemma

Step 4: Reducing $\|\alpha\|$.

By Step 3, we have $q \leq 16/\delta_2$, so $\delta_1 < 1/(2q)$. Recalling $|\alpha - a/q| \leq 1/(4qN)$, this gives

$$\|an/q\| < 1/q, \quad \text{for} \quad n \in S.$$

Thus $S \subset q\mathbb{Z} \cap [1, N]$.

Step 5: Finishing the proof.

Let $\theta = \alpha - a/q$. Since $S \subset q\mathbb{Z}$, we have $\|\theta n\| = \|\alpha n\|$ for $n \in S$. But $|\theta| \leq 1/(4Nq)$, so $\|\theta n\| = |\theta n|$ for all $n \leq N$. Thus

$$|\theta n| \leq \delta_1 \tag{1}$$

for $n \in S$. But since $|S| \geq \delta_2 N$ and $S \subset q\mathbb{Z}$, $\exists n_0 \in S$ such that $|n_0| \geq \delta_2 qN$. Choosing $n = n_0$ in (1), we get $|\theta| \leq \delta_1/(q\delta_2 N)$, so $\|\alpha q\| \leq \|\theta q\| \leq \delta_1/(\delta_2 N)$. □

Proof of Weyl sum estimate

We need one more ingredient.

Lemma 4.2.1

Let X be finite and $b : X \to \mathbb{C}$ such that $|b(x)| \leq 1$ for all $x \in X$. Suppose $|\sum_{x \in X} b(x)| \geq \varepsilon |X|$. Then there are $\geq \varepsilon |X|/2$ values of $x \in X$ for which $|b(x)| \geq \varepsilon |X|/2$.

Proof

Argue by contradiction. □

The proof of Prop. 4.2.1 is by induction, so we start with the simple case of $k = 1$.

Proof of Prop. 4.2.1 for $k = 1$. Let $P(x) = \alpha x + \beta$ be a linear polynomial. By the geometric sum formula, we have

$$|\sum_{x \in I} e(P(x))| = |\sum_{j=0}^{|I|-1} e(\alpha j)| = |\frac{1 - e(\alpha |I|)}{1 - e(\alpha)}| \leq 2/|1 - e(\alpha)| \ll 1/\|\alpha\|.$$

Hence, if the LHS is $> \delta L$, we must have $\|\alpha\| \ll \delta^{-1} L^{-1}$.

Proof of Weyl sum estimate

We use induction.The case $k = 1$ has been handled. Suppose that the case $k - 1$ has been handled, and consider case k.

Step 1: Square out and look at discrete derivatives.

By assumption we have

$$|\sum_{x \in I} e(P(x))|^2 \geq \delta^2 L^2, \text{ so } |\sum_{x,y \in I} e(P(x) - P(y))| \geq \delta^2 L^2.$$

Letting $h = y - x$ and introducing the discrete derivatives $\partial_h f(x) = f(x+h) - f(x)$,

$$|\sum_{|h| \leq L, x \in I_h} e(\partial_h P(x))| \geq \delta^2 L^2, \quad I_h = I \cap (I - h).$$

By the averaging lemma, this gives

$$\exists \geq \delta^2 L/6 \text{ values of } |h| \leq L \text{ s.t. } |\sum_{x \in I_h} e(\partial_h P(x))| \geq \delta^2 L/6.$$

Since $L > 100\delta^{-2}$, the contribution of $h = 0$ is small, so there are $\delta^2 L/18$ *positive* (or negative) h with this property.

Proof of Weyl sum estimate

Step 2: Applying the induction assumption.

Let H be the set of h of size at least $\delta^2 L/18$ from the previous slide. Note that crucially $\partial_h P(x) = k\alpha x^{k-1} + \ldots$ is a polynomial of degree $k-1$, so by induction

$$\forall h \in H \ \exists q_h \ll \delta^{-2C_{k-1}} \text{ s.t. } \|khq_h\alpha\| \ll \delta^{-2C_{k-1}} L^{-(k-1)}.$$

By pigeonholing, $\exists H' \subset H$ of size $\gg \delta^{2+2C_{k-1}} L$ such that $q_h := q'$ is constant for $h \in H'$.

Step 3: Applying Vinogradov's lemma.

Apply Vinogradov's lemma with $\alpha' = kq'\alpha$,
$\delta_1 = C_1 \delta^{-2C_{k-1}} L^{-(k-1)}$, $\delta_2 = c_2 \delta^{2+2C_{k-1}}$ (we have $\delta_2 > C\delta_1$, since $C_k > 2 + 4C_{k-1}$). Hence,

$$\exists q'' \ll \delta_2^{-1} \ll \delta^{-2-2C_{k-1}} \text{ s.t. } \|\alpha' q''\| \ll \delta_1/(\delta_2 L) \ll \delta^{-2-4C_{k-1}} L^{-k}.$$

Letting $q = kq'q''$ and recalling $C_k > 2 + 4C_{k-1}$, we are done. □