C3.10 Additive and Combinatorial NT
Lecture 6: Major arcs I

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Major arcs

Recall that the proof of

$$r_{k,s}(N) = \mathfrak{S}_{k,s}(N)N^{s/k-1} + o(N^{s/k-1})$$

has been reduced to

Prop. 3.2.1 (Major arcs)

Let $s \geq 2k+1$, $X = \{n^k : \; n \leq N^{1/k}\}$. Then,

$$\int_{\mathfrak{M}} \widehat{1_X}(\theta)^s e(N\theta)\, d\theta = \mathfrak{S}_{k,s}(N)N^{s/k-1} + o(N^{s/k-1}).$$

Definition

Set $\eta := 1/(10k)$. Define the *major arcs*

$$\mathfrak{M} = \bigcup_{\substack{q \leq N^\eta \\ a \in (\mathbb{Z}/q\mathbb{Z})^*}} \mathfrak{M}_{a,q}, \quad \mathfrak{M}_{a,q} := \{\theta \in \mathbb{T} : |\theta - \frac{a}{q}| \leq N^{-1+2\eta}\}.$$

Define the *minor arcs* $\mathfrak{m}$ to be $\mathbb{T} \setminus \mathfrak{M}$.

Single point on a major arc

The plan is to estimate $\widehat{1_X}(\theta)$
(1) at a single point of $\mathfrak{M}_{a,q}$;
(2) integrate over $\mathfrak{M}_{a,q}$;
(3) sum over a, q.

Prop. 6.1.1

For $\theta \in \mathfrak{M}_{a,q}$, we have

$$\widehat{1_X}(\theta) = G_{a,q} I(\theta - a/q) + O(N^{4\eta}),$$

where

$$I(t) = \int_0^{N^{1/k}} e(-tx^k)\, dx, \quad G_{a,q} = \frac{1}{q} \sum_{b \in \mathbb{Z}/q\mathbb{Z}} e(-ab^k/q).$$

Proof. We consider the case $a = 0$, $q = 1$. Then the claim is

$$\widehat{1_X}(\theta) = I(\theta) + O(N^{4\eta}).$$

Single point on a major arc

The claim is

$$\widehat{1_X}(\theta) = I(\theta) + O(N^{4\eta}).$$

It suffices to show that

$$e(-\theta n^k) = \int_n^{n+1} e(-\theta x^k)\,dx + O(N^{4\eta-1/k}), \quad |\theta| \le N^{2\eta-1}. \quad (1)$$

By the binomial theorem, for $x \in [n, n+1]$,

$$\theta x^k = \theta n^k + O(\theta n^{k-1}) = \theta n^k + O(N^{2\eta-1/k}). \quad (2)$$

Here $2\eta - 1/k < 4\eta - 1/k < 0$, as $\eta = 1/(10k)$.
For $|\varepsilon| \ll 1$ we have $e(\varepsilon) = 1 + O(\varepsilon)$. Now, substituting (2) to (1) the claim follows. $\square$

Contribution of a single major arc

Prop. 6.2.1

We have

$$\int_{\mathfrak{M}_{a,q}} \hat{1}_X(\theta)^s e(N\theta)d\theta = G_{a,q}^s e(\frac{aN}{q}) \int_{-\infty}^{\infty} I(t)^s e(Nt)dt + o(N^{\frac{s}{k}-1-2\eta})$$

Proof. By Prop. 6.1.1.,

$$\widehat{1_X}(\theta) = G_{a,q} I(\theta - \frac{a}{q}) + O(N^{4\eta})$$

$$\Longrightarrow \widehat{1_X}(\theta)^s = G_{a,q}^s I(\theta - \frac{a}{q})^s + O(N^{(s-1)/k+4\eta})$$

$$\Longrightarrow \int_{\mathfrak{M}_{a,q}} \widehat{1_X}(\theta)^s e(N\theta)\, d\theta = G_{a,q}^s e(\frac{aN}{q}) \int_{|t|\leq N^{2\eta-1}} I(t)^s e(Nt)\, dt + O(N^{\frac{(s-1)}{k}+6\eta-1}).$$

Note $\eta = 1/(10k) \Rightarrow (s-1)k/k + 6\eta - 1 < s/k - 1 - 2\eta$, so it suffices to extend the t integral.

Contribution of a single major arc

The remaining claim is that

$$\int_{|t|>N^{2\eta-1}} |I(t)|^s \, dt \ll N^{s/k-1-2\eta}.$$

From the previous lecture, we have $|I(t)| \ll |t|^{-1/k}$, so the LHS is

$$\int_{N^{2\eta-1}}^{\infty} |t|^{-s/k} \, dt \ll N^{s/k-1-2\eta}. \qquad \square$$

Contribution of all the major arcs

Prop. 6.4.1

We have

$$\int_{\mathfrak{M}} \hat{1}_X(\theta)^s e(N\theta)d\theta = \int_{-\infty}^{\infty} I(t)^s e(Nt)dt \sum_q A(q) + o(N^{\frac{s}{k}-1}), \quad (3)$$

where

$$A(q) := \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} G_{a,q}^s e(aN/q).$$

Proof. Note that the major arcs $\mathfrak{M}_{a,q}$ are all disjoint. Indeed, if $\theta \in \mathfrak{M}_{a,q} \cap \mathfrak{M}_{a',q'}$, then $|\theta - a/q|, |\theta - a'/q'| \leq N^{-1+2\eta}$, so $1/(qq') \leq |a/q - a'/q'| \leq 2N^{-1+2\eta}$, which contradicts $q, q' \leq N^{\eta}$. By Prop. 6.2.1., we obtain (3), but with the extra condition that $q \leq N^{\eta}$.

Contribution of all the major arcs

Recall the claim

$$\int_{\mathfrak{M}} \hat{1}_X(\theta)^s e(N\theta)d\theta = \int_{-\infty}^{\infty} I(t)^s e(Nt)dt \sum_q A(q) + o(N^{\frac{s}{k}-1}),$$

where

$$A(q) := \sum_{a\in(\mathbb{Z}/q\mathbb{Z})^*} G_{a,q}^s e(aN/q).$$

Note that by the bound $I(t) \ll |t|^{-1/k}$ we have

$$\left|\int_{-\infty}^{\infty} I(t)^s e(Nt)\, dt\right| \ll \int_{-\infty}^{\infty} \min\{N^{s/k}, |t|^{-s/k}\}\, dt \ll N^{s/k-1}.$$

To bound the contribution of $q > N^\eta$, it suffices to show that

$$\sum_{q>N^\eta} |A(q)| = o(1).$$

From Lecture 5, we have $|G_{a,q}| \ll q^{-1/k+o(1)}$, so $|A(q)| \ll q^{1-s/k+o(1)}$, so the claim follows. $\square$

Two identities

To complete the proof of Prop. 3.2.1, it suffices to show that

$$\int_{-\infty}^{\infty} I(t)^s e(Nt)dt \sum_q A(q) = \mathfrak{S}_{k,s}(N) N^{s/k-1}.$$

This in turn will follow from

$$\int_{-\infty}^{\infty} I(t)^s e(Nt)dt = \beta_\infty N^{s/k-1} = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} N^{s/k-1}$$

and

$$\sum_q A(q) = \prod_p \beta_p(N).$$

These are exact identities.
We postpone their proofs to the next lecture.