C3.10 Additive and Combinatorial NT
Lecture 7: Major arcs II

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Two identities

Recall that in the previous lecture we reduced the proof of

$$r_{k,s}(N) = \mathfrak{S}_{k,s}(N)N^{s/k-1} + o(N^{s/k-1})$$

to showing that

$$\int_{-\infty}^{\infty} I(t)^s e(Nt)dt = \beta_\infty N^{s/k-1} := \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} N^{s/k-1} \qquad (1)$$

and

$$\sum_q A(q) = \prod_p \beta_p(N), \quad A(q) := \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} G_{a,q}^s e(aN/q) \qquad (2)$$

for $s \geq 2k+1$. We call (1) an Archimedean identity and (2) a p-adic identity.

p-adic identity

Recall from the previous lecture that $|A(q)| \ll q^{1-s/k+o(1)} \ll q^{-1-1/(2k)}$, say. Hence, we have the "Euler product"

$$\sum_q A(q) = \prod_p (1 + A(p) + A(p^2) + \cdots).$$

Thus, it suffices to show that

$$\beta_p(N) = \sum_{j \geq 0} A(p^j).$$

Note that this involves showing that the limit $\beta_p(N)$ exists.

p-adic identity

Write $\beta_p(N) = \lim_{n\to\infty} \beta_{p,n}(N)$, where

$$\beta_{p,n}(N) = p^{-(s-1)n}|\{(x_1, \dots, x_s) \in (\mathbb{Z}/p^n\mathbb{Z})^s : \ x_1^k + \cdots + x_s^k = N\}|.$$

It suffices to show that

$$\beta_{p,n}(N) = \sum_{0\le j\le n} A(p^j),$$

since letting $n \to \infty$ and using the fact that $\sum_j |A(p^j)|$ converges, we obtain the identity (2).
Note that

$$\beta_{p,n}(N) = \sum_{a\in\mathbb{Z}/p^n\mathbb{Z}} G_{a,p^n}^s e(aN/p^n).$$

Now split the a sum according to the highest power p^{n-j} of p that divides a. Since $G_{a,p} = G_{a/p^{n-j},p^j}$, the contribution of a given j is $A(p^j)$, as wanted.

Archimedean identity

We are left with showing

$$\int_{-\infty}^{\infty} I(t)^s e(Nt)dt = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} N^{s/k-1}, \quad I(t) = \int_0^{N^{1/k}} e(-tx^k)\, dx$$

Making the substitution $x = N^{1/k} y^{1/k}$ in the definition of I and $t = u/N$ in the t integral above, the LHS becomes

$$\int_{-\infty}^{\infty} \left(\frac{1}{k} \int_0^1 y^{-1-1/k} e(-uy)\, dy \right)^s e(u)\, du.$$

The inner integral is the Fourier transform of
$f(y) := \frac{1}{k} y^{-1-1/k} 1_{[0,1]}(y)$.
We apply the identity

$$\int_{\mathbb{R}} \widehat{f}(u)^s e(u)\, du = f * \dots * f(1) = \frac{1}{k^s} \int (y_1 \cdots y_{s-1}(1 - \sum_{i \leq s-1} y_i))^{-1+1/k} dy_1 \dots dy_{s-1}.$$

Archimedean identity

On the other hand,

$$\Gamma(1/k)^s = \Big(\int_0^\infty e^{-v} v^{1/k-1} dv\Big)^s$$
$$= \int_0^\infty \cdots \int_0^\infty e^{-v_1-\cdots-v_s}(v_1\cdots v_s)^{1/k-1} dv_1\cdots dv_s.$$

Make the substitution $z = v_1+\cdots+v_s$, $y_i = v_i/z$, $i = 1,\ldots,s-1$. We have $\partial v_i/\partial z = y_i$, $i = 1,\ldots,s-1$, and $\partial v_i/\partial y_j = 1$ when $i = j$ and -1 when $i = s$. Therefore, the Jacobian of this transformation is z^{s-1}, so

$$\Gamma(1/k)^s = \int_{y_i\geq 0}\Big(\int_0^\infty e^{-z} z^{s/k-1} dz\Big)(y_1\cdots y_{s-1}(1-y_1-\cdots-y_{s-1}))^{1/k-1}$$
$$= \Gamma(s/k)\int_{y_i\geq 0}(y_1\cdots y_{s-1}(1-y_1-\cdots-y_{s-1}))^{1/k-1} dy_1\cdots dy_{s-1}.$$

Therefore,

$$(f*\cdots*f)(1) = k^{-s}\frac{\Gamma(1/k)^s}{\Gamma(s/k)} = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)},$$

concluding the proof.