# C3.10 Additive and Combinatorial NT
# Lecture 10: Plünnecke–Ruzsa inequality

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# Plünnecke–Ruzsa inequality

Our eventual goal is to prove

**Theorem (Freiman)**

Let $A \subset \mathbb{Z}$ satisfy $|A + A| \leq K|A|$. Then there exists a generalized arithmetic progression $P$ of dimension $\ll_K 1$ and of size $|P| \ll_K |A|$ such that $A \subset P$.

The following is an ingredient in its proof.

**Lemma (Plünnecke–Ruzsa inequality)**

Let $A$ and $B$ be additive sets with $|A + B| \leq K|A|$. Let $k, \ell$ be nonnegative integers. Then $|kB - \ell B| \leq K^{k+\ell}|A|$.

# A lemma on sumsets

Let $B \subset G$, $G$ abelian. Let $K \geq 0$, and define

$$\phi : 2^G \to \mathbb{R}, \quad \phi(A) := |A + B| - K|A|. \tag{1}$$

**Lemma (Lemma 10.3.1)**

We have the submodularity relation

$$\phi(A \cup A') + \phi(A \cap A') \leq \phi(A) + \phi(A').$$

# A lemma on sumsets

**Proof:** Write $\sigma(A) := A + B$. Note that

$$\sigma(A \cap A') \subset \sigma(A) \cap \sigma(A').$$

This implies

$$\begin{aligned} |\sigma(A) \cup \sigma(A')| &= |\sigma(A)| + |\sigma(A')| - |\sigma(A) \cap \sigma(A')| \\ &\leq |\sigma(A)| + |\sigma(A')| - |\sigma(A \cap A')|. \end{aligned}$$

Thus, $|\sigma|$ satisfies the submodularity property

$$|\sigma(A) \cap \sigma(A')| + |\sigma(A) \cap \sigma(A')| \leq |\sigma(A)| + |\sigma(A')|.$$

Combine this with

$$|A \cup A'| + |A \cap A'| = |A| + |A'|$$

to conclude the proof. $\square$

# A property of submodular functions

**Lemma 10.3.2**

Let $\phi$ be any submodular function. Let $A_1, \ldots, A_n$ be sets with the following property: $\phi(A_i) = 0$, and $\phi(Z_i) \geq 0$ for every subset $Z_i \subseteq A_i$. Then $\phi\big(\bigcup_{i=1}^{n} A_i\big) \leq 0$.

**Proof:** By submodularity, we have

$$\phi(A_i \cup S) \leq \phi(A_i \cup S) + \phi(A_i \cap S) \leq \phi(A_i) + \phi(S) = \phi(S).$$

We then conclude by induction on $n$. □

# Petridis's inequality

**Proposition (Petridis)**

Let $A, B$ be additive sets. Suppose $|A + B| = K|A|$ and $|Z + B| \geq K|Z|$ for all $Z \subseteq A$. Then, for any further set $S$ in the group, $|A + B + S| \leq K|A + S|$.

**Proof:** Apply Lemma 10.3.2 with $\phi(A) = |A + B| - K|A|$. Take $A_i = A + s_i$, where $S = \{s_i\}$. The hypotheses of Lemma 10.3.2 hold, since $\phi(Z) \geq 0$ for all $Z \subset A$ and $\phi(A_i) = 0$. Note that $\bigcup_{i=1}^{n} A_i = A + S$, so the lemma implies that $\phi(A + S) \leq 0$, i.e. $|A + B + S| \leq K|A + S|$. □

# Petridis's inequality

We will apply Petridis's inequality in the following form.

**Corollary 10.3.1**

Let $A, B$ be additive sets. Suppose that $|A + B| \leq K|A|$. Let $\emptyset \neq X \subseteq A$ be a set for which the ratio $|X + B|/|X|$ is minimal. Then for any set $S$ we have

$$|S + X + B| \leq K|S + X|.$$

**Proof:** Apply the previous proposition with $X$ in place of $A$. □

# Proof of Plünnecke–Ruzsa inequality

**Lemma 10.4.1**

Let $A$, $B$ be finite additive sets. Suppose $|A+B| \leq K|A|$. Then there exists $X \subset A$ such that $|X+kB| \leq K^k|X|$ for all $k \geq 0$.

**Proof:** Choose $X$ as the subset of $A$ for which the ratio $|X+B|/|X|$ is minimal. Petridis's inequality (Corollary 10.3.1) with $S=(k-1)B$ gives

$$|X+kB| = |X+(k-1)B+B| \leq K|X+(k-1)B|.$$

Now use induction on $k$. □

**Proof of Plünnecke–Ruzsa inequality:** Let $A, B$ be finite additive sets for which $|A+B| \leq K|A|$. Apply Ruzsa's triangle inequality with $(U,V,W)=(X,-kB,-\ell B)$ and Lemma 10.4.1 to obtain

$$|kB-\ell B|\ |X| \leq |X+kB|\cdot|X+\ell B| \leq K^{k+\ell}|X|^2.$$

Thus, since $X \subset A$, $|kB-\ell B| \leq K^{k+\ell}|X| \leq K^{k+\ell}|A|$. □