C3.10 Additive and Combinatorial NT
Lecture 13: Roth's theorem I

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Roth's theorem

Let $[N] := \{1, 2, \ldots, N\}$. Let us say that a three-term AP is nontrvial if it is of the form $x, x + d, x + 2d$ with $d \neq 0$.
Our goal in the next two lectures is to prove

Theorem (Roth)

Let $r_3(N)$ be the size of the largest subset of $[N]$ that is free of nontrivial 3-APs. Then $r_3(N) = o(N)$. In fact, more precisely $r_3(N) \ll N/(\log\log N)$.

- Roth's proof of his theorem introduced Fourier methods to additive combinatorics.
- Roth's theorem represents the first nontrivial case of Szemeredi's theorem: Any subset of $[N]$ that is free of non-trivial k-APs has size $o(N)$.
- Roth's bound has been improved using deep and insightful arguments, most recently to $r_3(N) \ll N/(\log N)^{1+\delta}$ $(\delta > 0)$ by Bloom–Sisask.

Density increment strategy

Roth's proof of his theorem splits into two parts:

- Proving a "density increment result",
- Deducing the theorem from it.

In what follows, $c > 0$ is a small constant and $C > 1$ is a large constant.

Proposition (Density increment)

Let $\alpha > 0$, $N \geq 1$, and let P be an arithmetic progression of size N. Then for any $A \subset P$ with $|A| \geq \alpha N$ and $N \geq C\alpha^{-C}$ at least one of the following holds.

1. A contains a nontrivial 3-AP,
2. There exists an AP $P' \subset P$ with $|P'| \geq N^c$ such that $|A \cap P'|/|P'| \geq \alpha + c\alpha^2$.

Density increment strategy

Proof that Proposition implies Roth's theorem: Let $A \subset [N]$ be free of 3-APs. We want to show $\alpha \leq C/\log\log N$.
Iterate the previous proposition to conclude that there exists a sequence $[N] = P_0 \supset P_1 \supset P_2 \supset \ldots$ of APs such that

- $|P_i| \geq |P_{i-1}|^c$,
- The relative densities $\alpha_i := |A \cap P_i|/|P_i|$ satisfy $\alpha_i \geq \alpha_{i-1} + c\alpha_{i-1}^2$.

This iteration can be continued as long as $|P_i| \geq C\alpha_i^{-C}$. Since $|P_i| \geq N^{c^i}$, $\alpha_i \geq \alpha$, we can continue it for $\geq (\log\log N - \log\log(C\alpha^{-C}))/\log(1/c)$ steps. However, after $1/(c\alpha)$ steps, the relative density α_i doubles, and after another $1/(2c\alpha)$ steps it quadruples, and so on. So in $\leq 2/(c\alpha)$ steps it exceeds one.
Hence, $2/(c\alpha) \geq (\log\log N - \log\log(C\alpha^{-C}))/\log(1/c)$, which implies $\alpha \geq C/\log\log N$. □

Applying Fourier analysis

We may assume that $P = [N]$ by an affine transformation.

Let f_A be the *balanced function* of A, $f_A = 1_A - \alpha 1_{[N]}$, $\alpha = |A|/N$.

Proposition (No 3-APs implies large Fourier coefficient)

Let $A \subset [N]$ satisfy $|A| \geq \alpha N$ and $N \geq 4/\alpha^2$. Then at least one of the following holds.

1. A contains a nontrivial 3-AP
2. There exists $\theta \in \mathbb{T}$ such that $|\widehat{f_A}(\theta)| \geq \alpha^2 N/28$.

Proof: For compactly supported functions, define the linear operator

$$T(f, g, h) := \sum_{x, d \in \mathbb{Z}} f(x) g(x+d) h(x+2d).$$

By the orthogonality relations, we have

$$T(f, g, h) = \int_0^1 \hat{f}(\theta) \hat{g}(-2\theta) \hat{h}(\theta)\, d\theta.$$

On the other hand,

$$T(1_A, 1_A, 1_A) = |A| + 2|\{\text{nontrivial 3-AP in } A\}|.$$

Applying Fourier analysis

We now conclude that if A contains no 3-APs, then $T(1_A, 1_A, 1_A) \leq \alpha N \leq \alpha^3 N/4$.
But by writing $1_A = f_A + \alpha 1_{[N]}$, we can split $T(1_A, 1_A, 1_A)$ as a sum of 8 terms

$$T(f_A, f_A, f_A) + \cdots + T(\alpha 1_{[N]}, \alpha 1_{[N]]}, \alpha 1_{[N]}).$$

But $T(\alpha 1_{[N]}, \alpha 1_{[N]]}, \alpha 1_{[N]}) \geq \alpha^3 N^2/2$, so one of the other 7 terms is $\geq \alpha^3 N^2/28$ in modulus.
Say $|T(f_1, f_2, f_3)| \geq \alpha^3 N^2/28$, $f_i \in \{f_A, \alpha 1_{[N]}\}$, and e.g. $f_1 = f_A$.
By the Fourier representation,

$$\int_0^1 \hat{f}_1(\theta)\hat{f}_2(-2\theta)\hat{f}_3(\theta)\, d\theta \geq \alpha^3 N^2/28.$$

Applying Fourier analysis

Thus, by Cauchy–Schwarz

$$\sup_\theta |\hat{f}_1(\theta)| \left(\int_0^1 |\hat{f}_2(\theta)|^2 \, d\theta \int_0^1 |\hat{f}_3(\theta)|^2 \, d\theta \right)^{1/2} \geq \alpha^3 N^2/28.$$

By Parseval, we get

$$\sup_\theta |\hat{f}_1(\theta)| \left(\sum_n |f_2(n)|^2 \sum_n |f_3(n)|^2 \right)^{1/2} \geq \alpha^3 N^2/28.$$

We have $\sum_n |f_i(n)|^2 \leq \alpha N$, so

$$\sup_\theta |\hat{f}_1(\theta)| = \sup_\theta |\widehat{f_A}(\theta)| \geq \alpha^2 N/28. \qquad \square$$

In the next lecture, we will use this Fourier proposition to conclude the proof.