C3.10 Additive and Combinatorial NT
Lecture 14: Roth's theorem II

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Roth's theorem

Recall that our goal is to prove

Theorem (Roth)

Let $r_3(N)$ be the size of the largest subset of $[N]$ that is free of nontrivial 3-APs. Then $r_3(N) = o(N)$. In fact, more precisely $r_3(N) \ll N/(\log\log N)$.

We reduced the proof to

Proposition (Density increment)

Let $\alpha > 0$, $N \geq 1$, and let P be an arithmetic progression of size N. Then for any $A \subset P$ with $|A| \geq \alpha N$ and $N \geq C\alpha^{-C}$ at least one of the following holds.

1. A contains a nontrivial 3-AP,
2. There exists an AP $P' \subset P$ with $|P'| \geq N^c$ such that $|A \cap P'|/|P'| \geq \alpha + c\alpha^2$.

Partitioning $[N]$ into subprogressions

We wish to deduce the density increment proposition from

Proposition (No 3-APs implies large Fourier coefficient)

Let $A \subset [N]$ satisfy $|A| \geq \alpha N$ and $N \geq 4/\alpha^2$. Then at least one of the following holds.

1. A contains a nontrivial 3-AP
2. There exists $\theta \in \mathbb{T}$ such that $|\widehat{f_A}(\theta)| \geq \alpha^2 N/28$.

We thus need to show that $|\widehat{f_A}(\theta)| \geq \alpha^2 N/28$ implies the existence of a progression $P \subset [N]$ of size $\geq N^c$ such that $|A \cap P|/|P| \geq \alpha + c\alpha^2$.

Lemma 9.3.1

There is a partition of $[N]$ into progressions P_i of length $\geq N^{1/5}$ such that

$$\sup_{x,y \in P_i} |e(\theta x) - e(\theta y)| \leq N^{-1/5} \quad \forall\, i.$$

Partitioning $[N]$ into subprogressions

Proof: Let $Q = \lfloor N^{1/2} \rfloor$. By Dirichlet's approximation theorem, there exist coprime a, q such that $q \leq Q$ and $|\theta - a/q| \leq 1/(qQ)$. Now, consider the progressions $\{n \leq N : \ n \equiv b \pmod q\}$, and split each of these further into progressions of the form $\{n \in I : \ n \equiv b \pmod q\}$, where $I \subset [N]$ has length $\in [qN^{1/5}, 2qN^{1/5}]$.
Then $|P_j| \geq N^{1/5}$ and if we denote $x_0 = \min P_j$, then for $x \in P_j$ we have

$$|e(\theta x) - e(\theta x_0)| = |e(\theta(x - x_0)) - 1| = |e((\theta - a/q)(x - x_0)) - 1|$$
$$\leq 2\pi|\theta - a/q||x - x_0| \leq 8\pi N^{1/5-1/2} < N^{-1/5}/2. \qquad \square$$

Finishing the proof

We are ready to prove the final ingredient.

Proposition (Large Fourier coefficient implies density increment)

Suppose that $|\widehat{f_A}(\theta)| \geq \alpha^2 N/28$, $N \geq (8/\alpha)^{10}$, and let P_i be as in Lemma 9.3.1. Then There exists i such that $|A \cap P_i| \geq (\alpha + \alpha^2/112)|P_i|$.

Proof: Recall that the P_i form a partition of $[N]$. Hence, by the triangle inequality,

$$\alpha^2 N/28 \leq \sup_\theta |\widehat{f_A}(\theta)| \leq \sum_i |\sum_{n \in P_i} f_A(n) e(\theta n)|.$$

We know that there exist some x_i such that $|e(\theta x) - e(\theta x_i)| \leq 2N^{-1/5}$ on P_i, so we get

$$\sum_i |\sum_{n \in P_i} f_A(n) e(\theta x_i)| \geq \alpha^2 N/28 - 2N^{1-1/5} \geq \alpha^2 N/56.$$

Finishing the proof

We have

$$\sum_i |\sum_{n\in P_i} f_A(n)| \geq \alpha^2 N/28 - 2N^{1/5} \geq \alpha^2 N/56$$

and

$$\sum_i \sum_{n\in P_i} f_A(n) \geq 0.$$

Hence, we obtain

$$\sum_i \left(|\sum_{n\in P_i} f_A(n)| + \sum_{n\in P_i} f_A(n) \right) \geq \alpha^2 N/56$$

Using $x + |x| = 2\max\{x, 0\}$, we see that for some i we have $\sum_{n\in P_i} f_A(n) = |A| - \alpha|P_i| \geq \alpha^2|P_i|/(2\cdot 56)$. $\square$