C3.10 Additive and Combinatorial NT
Lecture 15: The ternary Goldbach problem I

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The Goldbach conjectures

Conjecture (The ternary Goldbach conjecture)

Every odd integer $N \geqslant 7$ is the sum of three primes.

Conjecture (The binary Goldbach conjecture)

Every even integer $N \geqslant 4$ is the sum of two primes.

Since N odd $\implies$ $N - 3$ is even, the binary conjecture $\implies$ the ternary conjecture.
However, the binary Goldbach problem remains open, whereas the ternary Goldbach conjecture was solved by Vinogradov in 1937.

Theorem (Vinogradov's three primes theorem)

There exists a large N_0 such that every odd $N \geqslant N_0$ is the sum of three primes.

Vinogradov's three primes theorem

The proof actually gives an asymptotic for solutions to $N = p_1 + p_2 + p_3$. As usual, we count the primes with the weight

$$\Lambda(n) = \begin{cases} \log p, & n = p^k, \quad k \geqslant 1, \ p \text{ prime} \\ 0, & n \text{ not a prime power} \end{cases}$$

Theorem (Vinogradov's three primes theorem, quantitative)

Let

$$r(N) = \sum_{N = n_1 + n_2 + n_3} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3)$$

be the weighted count of solutions to $N = p_1 + p_2 + p_3$. Then

$$r(N) = \frac{1}{2}\mathfrak{S}(N) N^2 + O_C(N^2(\log N)^{-C})$$

for any $C \geqslant 1$, where the singular series is given by

$$\mathfrak{S}(N) = \prod_{p \mid N}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p \nmid N}\left(1 + \frac{1}{(p-1)^3}\right).$$

Vinogradov's theorem

Note that the quantitative version of Vinogradov's theorem easily implies the qualitative version: we have

$$r(N) \leqslant (\log N)^3 \sum_{N=p_1+p_2+p_3} 1 + O(N^{3/2}),$$

with the error term coming from higher prime powers, and for odd N we have

$$r(N) \gg N^2 \prod_{p|N} (1 - 1/(p-1)^2) \gg N^2.$$

Therefore, it suffices to prove the preceding theorem.

Two assumptions

We are going to assume two theorems from analytic NT as a given, as their proofs would take us too far from the main topic.

Theorem (Siegel–Walfisz theorem)

Let $A \geqslant 1$ and $x \geqslant 2$. Then for any coprime $a, q \geqslant 1$ we have

$$\sum_{n \leqslant x, n \equiv a \pmod q} \Lambda(n) = \frac{x}{\varphi(q)} + O_A(x/(\log x)^A).$$

Theorem (Davenport's bound)

Let $x \geqslant 2$, $A \geqslant 1$, and let $\alpha \in \mathbb{R}$ satisfy

$$\inf_{1 \leqslant q \leqslant (\log x)^A} \|q\alpha\|_{\mathbb{R}/\mathbb{Z}} \geqslant (\log x)^{100A}/x.$$

Then we have

$$|\sum_{n \leqslant x} \Lambda(n) e(\alpha n)| \ll_A x(\log x)^{-A}.$$

Major and minor arcs

By the orthogonality relations, we have

$$r(N) = \int_0^1 S(\alpha)^3 e(-N\alpha),$$

where

$$S(\alpha) = \sum_{n \leqslant N} \Lambda(n) e(\alpha n).$$

Let $A \geqslant 1$ be a large enough constant, and let $P = (\log N)^A$, $Q = N/(\log N)^{2A}$. Define the major arcs as

$$\mathfrak{M} = \bigcup_{1 \leqslant q \leqslant P} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \mathfrak{M}_{a,q}, \quad \mathfrak{M}_{a,q} = \{\alpha \in \mathbb{T} : \ \|\alpha - a/q\|_{\mathbb{R}/\mathbb{Z}} \leqslant 1/(qQ)\}$$

and the minor arcs as

$$\mathfrak{m} = \mathbb{T} \setminus \mathfrak{M}.$$

Note that the major arcs $\mathfrak{M}_{a,q}$ are pairwise disjoint, since if $a/q, a'/q'$ are the midpoints of two different major arcs, we have $\|a/q - a'/q'\|_{\mathbb{R}/\mathbb{Z}} \geqslant 1/(qq') \geqslant 1/P^2 > 10/Q$.

Minor arc case

Using Parseval's identity, we estimate

$$\begin{aligned}\left|\int_{\mathfrak{m}} S(\alpha)^3 e(-N\alpha)\,d\alpha\right| &\leqslant \sup_{\alpha\in\mathfrak{m}} |S(\alpha)| \int_0^1 |S(\alpha)|^2\,d\alpha \\ &= \sup_{\mathfrak{m}} |S(\alpha)| \cdot \sum_{n\leqslant N} \Lambda(n)^2 \\ &\ll \sup_{\mathfrak{m}} |S(\alpha)| \cdot N(\log N),\end{aligned}$$

where we used the prime number theorem in the last step. Now, since $\alpha \in \mathfrak{m} \implies \|q\alpha\| \geqslant 1/Q = (\log N)^{2A}/x \,\forall\, q \leqslant (\log N)^A$, Davenport's theorem gives

$$\sup_{\mathfrak{m}} |S(\alpha)| \ll N(\log N)^{-A/100}.$$

Combining this with the above and taking $A = 200C$, say, we see that the minor arc contribution is $\ll_C N^2/(\log N)^C$.

We are left with the major arc case, where we need

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\,d\alpha = \frac{1}{2}\mathfrak{S}(N)N^2 + O_C(N^2(\log N)^{-C}).$$

Dirichlet characters

A *character* of an abelian group G is simply a homomorphism $\chi : G \to \mathbb{C} \setminus \{0\}$. Consider the case $G = \mathbb{Z}/q\mathbb{Z}$. In this case, we can extend the character from $\mathbb{Z}/q\mathbb{Z}$ to $\mathbb{Z}$ by periodicity.

Definition

We say that $\chi : \mathbb{Z} \to \mathbb{C}$ is a *Dirichlet character* modulo q if $\chi(n) = 0$ for $(n, q) = 1$, and χ is q-periodic, and $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$.

The function $\chi_0(n) = 1_{(n,q)=1}$ is clearly a Dirichlet character; it is called the *principal character*.
Example: A character $\pmod 5$: $\chi(0) = 0$, $\chi(1) = 1, \chi(2) = i$, $\chi(3) = -i$, $\chi(4) = -1$.

Character orthogonality

Lemma

1. Let $q \geqslant 1$ and let a, b be coprime to q. Then

$$\sum_{\chi \pmod q} \chi(a)\overline{\chi}(b) = \varphi(q) 1_{a \equiv b \pmod q}.$$

2. Let $q \geqslant 1$, and let $\chi_1, \chi_2 \pmod q$ be Dirichlet characters. Let a be coprime to q. Then

$$\sum_{a \pmod q} \chi_1(a)\overline{\chi_2}(a) = \varphi(q) 1_{\chi_1 = \chi_2}.$$

Proof: The first claim follows from the facts that (i) $\{\chi \pmod q\}$ is a group, (ii) $|\{\chi \pmod q\}| = \varphi(q)$, (iii) if $a \not\equiv 1 \pmod q$, then $\chi(a) \neq 1$ for some $\chi \pmod q$. These facts can be verified relatively easily using primitive roots.
The second claim is proved very similarly. $\square$

Siegel–Walfisz again

We can reformulate the Siegel–Walfisz in terms of characters.

Theorem (Siegel–Walfisz theorem, characters)

Let $A \geqslant 1$, $x \geqslant 2$. For any $\chi \pmod q$ with $q \leqslant (\log x)^A$ we have

$$\psi(x,\chi) := \sum_{n \leqslant x} \Lambda(n)\chi(n) = x1_{\chi=\chi_0} + O_A(x(\log x)^{-A}).$$

Proof: Applying Siegel–Walfisz, we have

$$\begin{aligned}\sum_{n\leqslant x}\Lambda(n)\chi(n) &= \sum_{a \pmod q} \chi(a) \sum_{\substack{n\leqslant x \\ n\equiv a \pmod q}} \Lambda(n) \\ &= \sum_{a \pmod q} \chi(a)\frac{x}{\varphi(q)} + O_A\Big(\frac{qx}{(\log x)^{2A}}\Big) \\ &= x1_{\chi=\chi_0} + O\Big(\frac{x}{(\log x)^A}\Big),\end{aligned}$$

where the last step used the orthogonality of characters. $\square$

Fourier expansion of exponential

Lemma

For $n, q \geqslant 1$ and $(n, q) = 1$, we have

$$e\left(\frac{n}{q}\right) = \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \tau(\overline{\chi})\chi(n), \tag{1}$$

where

$$\tau(\chi) = \sum_{1 \leqslant n \leqslant q} \chi(n) e\left(-\frac{n}{q}\right). \tag{2}$$

Proof. This follows by substituting (2) into (1) and using the orthogonality relations. $\square$