C3.10 Additive and Combinatorial NT
Lecture 16: The ternary Goldbach problem II

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Recap

To finish the proof of Vinogradov's theorem, we must show that

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\,d\alpha = \frac{1}{2}\mathfrak{S}(N)N^2 + O_C(N^2(\log N)^{-C}),$$

where

$$\mathfrak{S}(N) = \prod_{p|N}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p\nmid N}\left(1 + \frac{1}{(p-1)^3}\right).$$

Here the major arcs are given by

$$\mathfrak{M} = \bigcup_{1\leqslant q\leqslant P}\ \bigcup_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} \mathfrak{M}_{a,q}, \quad \mathfrak{M}_{a,q} = \{\alpha\in\mathbb{T} :\ \|\alpha - a/q\|_{\mathbb{R}/\mathbb{Z}} \leqslant 1/(qQ)\}$$

and $P = (\log N)^A,\ Q = N/(\log N)^{2A}$.

Midpoint of a major arc

We first consider $S(a/q)$ for $1 \leqslant a \leqslant q \leqslant P$.
By the character expansion of $e(an/q)$ and Siegel–Walfisz,

$$\begin{aligned}
S(\frac{a}{q}) &= \sum_{n \leqslant N} \Lambda(n) e(\frac{an}{q}) \\
&= \sum_{\substack{n \leqslant N \\ (n,q)=1}} \Lambda(n) e(\frac{an}{q}) + O((\log q)^2) \\
&= \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \tau(\overline{\chi}) \chi(a) \sum_{n \leqslant N} \Lambda(n) \chi(n) \\
&= \sum_{\chi \pmod q} \tau(\overline{\chi}) \chi(a) \frac{N}{\varphi(q)} 1_{\chi=\chi_0} + O(\frac{N}{(\log N)^{10A}}) \\
&= \frac{\tau(\chi_0)}{\varphi(q)} N + O(\frac{N}{(\log N)^{10A}}).
\end{aligned}$$

Lastly, note that $\tau(\chi_0) = \mu(q)$ by Möbius inversion. Thus

$$S(a/q) = \frac{\mu(q)}{\varphi(q)} N + O(N(\log N)^{-10A}).$$

Arbitrary point on a major arc

To generalize this to the case $\alpha \in \mathfrak{M}_{a,q}$, we apply partial summation (C3.8 course). Let $\alpha = a/q + \beta$. Partial summation and the bound on the previous slide give

$$S(\frac{a}{q} + \beta) = e(\beta N) \sum_{n \leqslant N} \Lambda(n) e(\frac{an}{q}) - 2\pi i \beta \int_1^N \sum_{n \leqslant t} \Lambda(n) e(\frac{an}{q}) e(\beta t)\, dt$$
$$= \frac{\mu(q)}{\varphi(q)} \left(N e(N\beta) - 2\pi i \beta \int_1^N t e(\beta t)\, dt \right)$$
$$+ O((1 + |\beta| N) N / (\log N)^{10A}).$$

Again applying partial summation, and denoting

$$T(\beta) = \sum_{n \leqslant N} e(\beta n),$$

we obtain

$$S(\frac{a}{q} + \beta) = \frac{\mu(q)}{\varphi(q)} T(\beta) + \frac{N}{(\log N)^{8A}},$$

say, since $|\beta| N \leqslant N/Q \leqslant (\log N)^{2A}$.

It is easy to see that the error term is small, even after summing over all the major arcs. It suffices to prove that

$$\sum_{\substack{q\leqslant Q\\ 1\leqslant a\leqslant q\\ (a,q)=1}} \int_{a/q-1/Q}^{a/q+1/Q} \frac{\mu(q)}{\varphi(q)^3} T(\alpha)^3 e(-N\alpha)\, d\alpha = \frac{N^2}{2}\mathfrak{S}(N) + O_A(\frac{N^2}{(\log N)^{A/2}})$$

We first consider the integral

$$I(\beta) = \int_{[-1/Q,1/Q]} T(\beta)^3 e(-N\beta)\, d\beta.$$

We have $|T(\beta)| \ll 1/\|\beta\|_{\mathbb{R}/\mathbb{Z}} \ll Q$ for $\beta \in \mathrm{T} \setminus [-1/Q, 1/Q]$, so

$$I(\beta) = \int_0^1 T(\beta)^3 e(-N\beta)\, d\beta + O(\int_{1/Q}^1 \frac{d\beta}{\beta^3}) = \int_0^1 T(\beta)^3 e(-N\beta)\, d\beta + O(Q^2).$$

The integral is counting solutions to $N = n_1 + n_2 + n_3$ with $1 \leqslant n_i \leqslant N$, and their number is clearly $\frac{1}{2}N^2 + O(N)$. Hence,

$$I(\beta) = N^2/2 + O(Q^2).$$

Summing over major arcs

Using $I(\beta) = N^2/2 + O(Q^2)$, and making a change of variables, the sum that we are considering becomes

$$\sum_{q \leqslant Q} \sum_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \frac{\mu(q)}{\varphi(q)^3} e(\frac{aN}{q}) \cdot \frac{1}{2} N^2 + O(P^2 Q^2)$$

Here the error term is $\ll N^2 (\log N)^{-2A}$.
We now define the *Ramanujan sum*

$$c_q(N) = \sum_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} e\left(\frac{aN}{q}\right).$$

With this notation, we can write the main term above as

$$\sum_{q \leqslant Q} \frac{\mu(q)}{\varphi(q)^3} c_q(N) \cdot \frac{1}{2} N^2. \tag{1}$$

Ramanujan sums

Crudely estimating $|c_q(N)| \leqslant \varphi(q)$, we have

$$\left| \sum_{q>P} \frac{\mu(q)}{\varphi(q)^3} c_q(N) \right| \leqslant \sum_{q>P} \frac{1}{\varphi(q)^2} \ll P^{-0.9},$$

since $\varphi(q) \gg q^{0.99}$. Therefore, we can complete the q sum to reduce matters to

$$\sum_{q \geqslant 1} \frac{\mu(q)}{\varphi(q)^3} c_q(N) = \mathfrak{S}(N).$$

We need one more lemma before we can prove this identity.

Lemma

Let $n \geqslant 1$. Then

1. $q \mapsto c_q(n)$ is multiplicative.
2. For primes p, we have $c_p(n) = p1_{p|n} - 1$.

Proof. For proving (i), let $q = q_1 q_2$ with $(q_1, q_2) = 1$. Using the fact that every reduced reside class $\pmod{q_1 q_2}$ is uniquely of the form $q_1 x + q_2 y$ with $x \in (\mathbb{Z}/q_2\mathbb{Z})^*$ and $y \in (\mathbb{Z}/q_1\mathbb{Z})^*$, we see that

$$c_q(n) = \sum_{\substack{1\leqslant x\leqslant q_2 \\ (x,q_2)=1}} \sum_{\substack{1\leqslant y\leqslant q_1 \\ (y,q_1)=1}} e\left(\frac{(q_1 x + q_2 y)n}{q}\right) = c_{q_1}(n)c_{q_2}(n).$$

For proving (ii), we use the orthogonality relations to write

$$c_p(n) = \sum_{1\leqslant a\leqslant p} e(an/p) - 1 = p1_{p|n} - 1.$$

This is the desired claim. □

Finishing the proof

By the previous lemma $f(q) = \mu(q)c_q(N)/\varphi(q)^3$ is multiplicative with $|f(q)| \ll q^{-1.9}$. Hence, we have the Euler product

$$\sum_{q \geqslant 1} f(q) = \prod_p (1 + f(q)) = \prod_p \left(1 - \frac{p 1_{p|N} - 1}{(p-1)^3}\right),$$

which can be verified by truncating the product on the right, expanding out, and estimating the error terms from the truncation trivially.

But since the product on the right-hand side is precisely $\mathfrak{S}(N)$ (as is seen by separating the primes $p \mid N$), we obtain

$$\sum_{q \geqslant 1} \frac{\mu(q)}{\varphi(q)^3} c_q(N) = \mathfrak{S}(N).$$

This finishes the proof of Vinogradov's theorem.