

**C3.10 Additive and Combinatorial Number Theory, Michaelmas  
2020 Sheet 0**

This example sheet will not be marked. It provides some exercises on the basic notation for the course, as well as on the Fourier transform.

**Question 1.** Which of the following statements are true, as  $x \rightarrow \infty$ ?

- (i)  $100x + 1000 = O(x)$ ;
- (ii)  $\sqrt{x} = o(x/\log x)$ ;
- (iii)  $ax^2 + bx + c \ll_{a,b,c} x^2$ ;
- (iv)  $e^{\sqrt{\log x}} = x^{o(1)}$ ;
- (v)  $x^{1/\log \log x} = O(e^{(\log x)^{2/3}})$ ;
- (vi)  $\log^A x \ll_A x^{1/10}$  with  $A \geq 1$ .

**Question 2.** Give an expression for the Fourier transform  $\hat{f}(\xi)$ , where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is the function taking value 1 for  $|x| \leq 1$  and 0 otherwise. Show that  $|f(\xi)| \ll |\xi|^{-1}$  as  $|\xi| \rightarrow \infty$ .

**Question 3.** With  $f$  as in the previous question, draw a picture of the convolution  $g = f * f$ . Show that  $\int_{-\infty}^{\infty} |\hat{g}(\xi)| d\xi < \infty$ .

**Question 4.** Let  $A \subset \mathbb{Z}/q\mathbb{Z}$ , and suppose that  $|A| = \alpha q$ . Let  $1_A$  be the characteristic function of  $A$ , that is to say the function taking value 1 on  $A$ , and 0 outside  $A$ . Show that  $|\hat{1}_A(r)| \leq \alpha$  for all  $r$ . Show additionally that the number of  $r \in \mathbb{Z}/q\mathbb{Z}$  with  $|\hat{1}_A(r)| \geq \eta\alpha$  is at most  $\eta^{-2}\alpha^{-1}$ .

**Question 5.** Let  $f : \mathbb{Z} \rightarrow \mathbb{C}$  be a compactly supported function. Show that  $\int_0^1 |\hat{f}(\theta)|^4 d\theta$  is the sum of  $f(n_1)f(n_2)\overline{f(n_3)}\overline{f(n_4)}$  over all quadruples  $(n_1, n_2, n_3, n_4)$  with  $n_1 + n_2 = n_3 + n_4$ .

**Question 6.** Let  $q$  be large, and let  $A \subset \mathbb{Z}/q\mathbb{Z}$  be the set  $\{1, \dots, \lfloor q/10 \rfloor\}$ . Show that  $|\hat{1}_A(1)| \geq 1/100$ .

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