C3.10 Additive and Combinatorial Number Theory, Michaelmas 2020 Sheet 0

This example sheet will not be marked. It provides some exercises on the basic notation for the course, as well as on the Fourier transform.

Question 1. Which of the following statements are true, as $x \to \infty$?

(i) $100x + 1000 = O(x)$;

(ii) $\sqrt{x} = o(x/\log x)$;

(iii) $ax^2 + bx + c \ll_{a,b,c} x^2$;

(iv) $e^{\sqrt{\log x}} = x^{o(1)}$;

(v) $x^{1/\log\log x} = O(e^{(\log x)^{2/3}})$;

(vi) $\log^A x \ll_A x^{1/10}$ with $A \geqslant 1$.

Solution. (i) true, the LHS is $\leqslant 101x$ for large enough x.

(ii) true, $\log x$ is much smaller than $\sqrt{x}$. The easiest way to see this is probably to take exponentials, thus one needs to show that t is much smaller than $e^{t/2}$, for large t. But this is obvious by the Taylor expansion of $e^{t/2}$.

(iii) true. In actual fact, this notation $\ll_{a,b,c}$ may not appear on the course, but it means that the constant in the $\ll$ notation may depend on a, b, c.

(iv) true. Taking logs, we just need to show that $\sqrt{\log x} \leqslant \varepsilon \log x$ for x sufficiently large, which is clear.

(v) false. Taking logs, the LHS becomes $\log x/\log\log x$, while the RHS is $O((\log x)^{2/3})$, which is much smaller since $(\log x)^{1/3}$ is much bigger than $\log\log x$ (if you like, by similar reasoning to (ii)).

(vi) true. We can take Ath roots and it amounts to showing that $\log x \ll_A x^{1/10A}$ or, writing $t = \log x$, that $t \ll_A e^{t/10A}$. But this follows using a Taylor series expansion as in (ii).

Question 2. Give an expression for the Fourier transform $\hat{f}(\xi)$, where $f : \mathbb{R} \to \mathbb{R}$ is the function taking value 1 for $|x| \leqslant 1$ and 0 otherwise. Show that $|f(\xi)| \ll |\xi|^{-1}$ as $|\xi| \to \infty$.

Solution. By definition,

$$\hat{f}(\xi) = \int_{-1}^{1} e(-x\xi)dx.$$

The antiderivate of $e(-x\xi) = e^{-2\pi i x\xi}$ is $\frac{1}{2\pi i\xi}e(-x\xi)$, so

$$\hat{f}(\xi) = -\frac{1}{2\pi i\xi}(e(-\xi) - e(\xi)).$$

If you like, you can write this in terms of sines:

$$e(-\xi) - e(\xi) = -2\sin 2\pi\xi$$

and so

$$\hat{f}(\xi) = \frac{\sin 2\pi\xi}{\pi i \xi}.$$

With either expression, the stated bound is fairly obvious.

Question 3. With f as in the previous question, draw a picture of the convolution $g = f * f$. Show that $\int_{-\infty}^{\infty} |\hat{g}(\xi)| d\xi < \infty$.

Solution. The graph of g is a tent function taking the value 0 at ± 2 and 2 at 0. We have $\hat{g}(\xi) = \hat{f}(\xi)^2$, and so by the previous question $|\hat{g}(\xi)| \ll |\xi|^{-2}$ as $|\xi| \to \infty$. The RHS is an integrable function.

Question 4. Let $A \subset \mathbb{Z}/q\mathbb{Z}$, and suppose that $|A| = \alpha q$. Let 1_A be the characteristic function of A, that is to say the function taking value 1 on A, and 0 outside A. Show that $|\hat{1}_A(r)| \leqslant \alpha$ for all r. Show additionally that the number of $r \in \mathbb{Z}/q\mathbb{Z}$ with $|\hat{1}_A(r)| \geqslant \eta\alpha$ is at most $\eta^{-2}\alpha^{-1}$.

Solution. The definition of the discrete Fourier transform gives

$$\hat{1}_A(r) = \frac{1}{q}\sum_{x \in A} e(-rx/q).$$

Since $|e(-rx/q)| \leqslant 1$ always, the bound $|\hat{1}_A(r)| \leqslant \alpha = |A|/q$ is immediate. For the second part, we use Parseval's identity, which tells us that

$$\sum_r |\hat{1}_A(r)|^2 = \frac{1}{q}\sum_x 1_A(x)^2 = \alpha.$$

If R is the set on which $|\hat{1}_A(r)| \geqslant \eta\alpha$, then the LHS of the above equation is at least $|R|\eta^2\alpha^2$. The claimed bound follows immediately.

Question 5. Let $f : \mathbb{Z} \to \mathbb{C}$ be a compactly supported function. Show that $\int_0^1 |\hat{f}(\theta)|^4 d\theta$ is the sum of $f(n_1)f(n_2)\overline{f(n_3)f(n_4)}$ over all quadruples (n_1, n_2, n_3, n_4) with $n_1 + n_2 = n_3 + n_4$.

Solution. Substitute in the definition of Fourier transform in the integral. It becomes

$$\int_0^1 \sum_{n_1,n_2,n_3,n_4} f(n_1)f(n_2)\overline{f(n_3)f(n_4)}e(-(n_1+n_2-n_3-n_4)\theta)d\theta.$$

Now interchange the order of summation and integration, and apply the orthogonality relation to conclude that the inner integral is nonzero if and only if $n_1 + n_2 - n_3 - n_4 = 0$.

Question 6. Let q be large, and let $A \subset \mathbb{Z}/q\mathbb{Z}$ be the set $\{1, \ldots, \lfloor q/10 \rfloor\}$. Let 1_A be the characteristic function of A. Show that $|\hat{1}_A(1)| \geqslant 1/100$.

Solution. By definition,

$$\hat{1}_A(1) = \frac{1}{q} \sum_{x \leqslant q/10} e(-x/q).$$

Taking real parts, and noting that $\lfloor q/10 \rfloor \geqslant q/11$ for q large enough, we see that

$$|\hat{1}_A(1)| \geqslant \frac{1}{q} \sum_{x \leqslant q/10} \cos(2\pi x/q) \geqslant \frac{1}{q} \frac{q}{11} \cos(\frac{2\pi}{10}).$$

This is considerably stronger than the stated result.

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