C3.10 Additive and Combinatorial Number Theory, Michaelmas 2020 Exercises 1

Question 1. Let $p = 5101$ (this is a prime). Starting from the observation that $2p = 101^2 + 1$, find a representation of p as a sum of two squares.

Question 2. Show that if $n = 4^a(8k+7)$ then n is *not* a sum of three squares.

Question 3. Show that a number is the sum of two squares if and only if all primes that are $3 \pmod 4$ occur in the prime factorisation of n with an even exponent.

Question 4. As in lectures, write $r_{k,s}(N)$ for the number of representations of N as a sum of s kth powers. Prove from first principles that there are positive constants c, C (depending on k, s but not on X) such that for sufficiently large X we have

$$cX^{s/k} \leqslant \sum_{N \leqslant X} r_{k,s}(N) \leqslant CX^{s/k}.$$

Explain why it follows that $G(k) \geqslant k$. Can one have $G(k) = k$ for $k \geqslant 2$?

Question 5. Show that every positive integer is the sum of at most 53 fourth powers. *Hint: you may wish to consider the identity*

$$6(\sum_{i=1}^{4} a_i^2)^2 = \sum_{1 \leqslant i < j \leqslant 4} ((a_i + a_j)^4 + (a_i - a_j)^4).$$

Question 6. By considering numbers of the form $31 \cdot 2^{4k}$, or otherwise, show that $G(4) \geqslant 16$.

Question 7. Does $r_{2,4}(N) \to \infty$ as $N \to \infty$?

Question 8. Show that every $n \leqslant 3^{100}$ is the sum of at most 20 cubes. (You should be able to do this with just a pocket calculator.)

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