

**C3.10 Additive and Combinatorial Number Theory, Michaelmas
2020 Exercises 3**

Question 1. Let $S \subset \mathbb{Z}$ be a finite set, and let a_1, \dots, a_k be nonzero integers. Give a formula for the number of tuples $(x_1, \dots, x_k) \in S$ such that $a_1x_1 + \dots + a_kx_k = 0$ in terms of the Fourier transform $\hat{1}_S$.

Question 2. By considering points on spheres, or otherwise, show that there is a set $A \subset [N]^d$, free of 3-term arithmetic progressions, with $|A| \geq \frac{1}{d}N^{d-2}$.

By considering maps of the form $(x_1, \dots, x_d) \rightarrow x_1 + Lx_2 + \dots + L^{d-1}x_d$ for appropriate M , show that if $\varepsilon > 0$, then for infinitely many values of M there is a set $A \subset [M]$, free of 3-term progressions, with $|A| \geq M^{1-\varepsilon}$.

Question 3. Let $G = \mathbb{F}_2^n$, and let e_1, \dots, e_n be basis vectors for this vector space. Let Σ_r be the Hamming ball of radius r , that is to say the set of sums of at most r of the basis vectors e_i .

Show that G may be covered by at most

$$\frac{2^n}{1 + \binom{n}{1} + \dots + \binom{n}{\lfloor n/2 \rfloor}}$$

translates of Σ_r .

Question 4. Suppose that $A \subset \mathbb{F}_2^n$ has doubling constant at most K . Show that $3A \subset 2A + X$, where $|X| \leq K^4$. Deduce that the subspace generated by A has size at most $K2^{K^4}|A|$.

Question 5. Give an example of a set A of integers with $|A + A| < |A - A|$. Is it true that there is an absolute constant K such that $|A - A| \leq K|A + A|$ for all sets of integers?

Question 6. Suppose that A is a finite set of integers with $|2A| \geq 100|A|$. Is it true that $|3A| \geq 1000|A|$?

Question 7. Let p be a prime, and suppose that $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$ are nonempty sets containing 0.

- (i) For $e \in \mathbb{Z}/p\mathbb{Z}$, write $A_e := A \cup (B + e)$ and $B_e := B \cap (A - e)$. Show that $|A_e| + |B_e| = |A| + |B|$ and that $A_e + B_e \subseteq A + B$.
- (ii) Suppose that $A \neq \mathbb{Z}/p\mathbb{Z}$. Show that there is some $e \in A$ such that $|A_e| > |A|$.
- (iii) Prove the Cauchy–Davenport Theorem: if $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$ are nonempty then $|A + B| \geq \min(|A| + |B| - 1, p)$.

Question 8. Let $\alpha > 0$. Show that there is a constant $\alpha' > 0$ such that the following is true: if $A \subseteq [N]$ is a set of size αN , with $N \geq N_0(\alpha)$, then A contains at least $\alpha' N^2$ 3-term APs. *Note: you may of course take Roth's theorem as a given here.*

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