C3.10 Additive and Combinatorial Number Theory, Michaelmas 2020
Exercises 4

Comment. This sheet is loosely based around proving the following result of Furstenberg and Sárközy.

Theorem 1. Let $\alpha > 0$. Suppose that $N > N_0(\alpha)$. Then any set $A \subset [N]$ with $|A| \geqslant \alpha N$ contains two different elements a, a' differing by a square.

I have divided the proof of the theorem up into exercises which all have something to do with other parts of the course, and which can be attempted more-or-less independently of one another.

The first set of questions concern the following theorem.

Theorem 2. We have

$$\lim_{N\to\infty} \sup_{\theta\in\mathbb{R}} \inf_{1\leqslant n\leqslant N} \|n^2\theta\|_{\mathbb{R}/\mathbb{Z}} = 0.$$

Statements like this are a little hard to parse, so let us reflect on the meaning: given $\theta \in \mathbb{R}$ and $\varepsilon > 0$, we can find $n \leqslant O_\varepsilon(1)$ such that $\|n^2\theta\|_{\mathbb{R}/\mathbb{Z}} \leqslant \varepsilon$, where the $O_\varepsilon(1)$ is uniform in θ.

Question 1. Let $\varepsilon \in (0, 1/2)$. Show that there exists a 1-periodic trigonometric polynomial $T(x)$ (depending on ε) such that for $|x| \leqslant 1/2$ we have $1_{[-\varepsilon,\varepsilon]}(x) \geqslant T(x)$ and such that $\int_{-1/2}^{1/2} T(x)\,dx > 0$. *Hint: the inequality* $1_{[-\varepsilon,\varepsilon]}(x) \geqslant \cos^{2k}(\pi x) - \cos^{2k}(\pi\varepsilon)$ *may be helpful for appropriately chosen* k.

Question 2. Suppose that there is no $n \leqslant N$ such that $\|n^2\theta\|_{\mathbb{R}/\mathbb{Z}} \leqslant \varepsilon$.

(i) Using the result of Question 1, or otherwise, show that there is some $m = O_\varepsilon(1)$, $m \neq 0$, such that

$$|\sum_{n\leqslant N} e(m\theta n^2)| \gg N.$$

(the implied constants here should be uniform in θ).

(ii) Using an appropriate result from the course, show that there is some nonzero $q = O_\varepsilon(1)$ such that $\|q\theta\|_{\mathbb{R}/\mathbb{Z}} \ll_\varepsilon N^{-2}$.

(iii) Prove Theorem 2.

Question 3. Sketch a proof of the following result. There is a function $\omega(N) \to \infty$ with the following property. For any $N \geqslant 1$ there is a partition $[N] = P_1 \cup \cdots \cup P_m$ into progressions with square common difference, with $|P_i| \geqslant \omega(N)$ for all i, and such that $\sup_{x,y\in P_i} |e(\theta x) - e(\theta y)| \leqslant \omega(N)^{-1}$ for all i.

Given two functions $f_1, f_2 : [N] \to \mathbb{R}$, define

$$T(f_1, f_2) := \sum_{x,d} f_1(x) f_2(x+d) 1_X(d),$$

where $X = \{n^2 : n \leqslant N^{1/2}\}$ (as in the course, specialised to $k = 2$).

Question 4. Write an expression for $T(f_1, f_2)$ in terms of the Fourier transforms of f_1, f_2 and 1_X.

Write $f_A = 1_A - \alpha 1_{[N]}$ for the balanced function of A.

Question 5. Suppose that A does not have any pair of elements differing by a square. Show that there are two functions g_1, g_2 bounded in modulus by 1, at least one of which is f_A, such that $|T(g_1, g_2)| \gg \alpha^2 N^{3/2}$.

Question 6. Using any results from the course that you like, explain why there is a positive integer s such that

$$\int_0^1 |\hat{1}_X(\theta)|^{2s} d\theta \ll N^{s-1}.$$

Question 7. Suppose that $g_1, g_2 : [N] \to \mathbb{R}$ are two functions bounded by 1 in modulus. Suppose that $T(g_1, g_2) \geqslant \delta N^{3/2}$. Show that for $i = 1, 2$ we have $\sup_\theta |\hat{g}_i(\theta)| \gg_\delta N$. *Hint:* you may wish to use Hölder's inequality, which states that

$$\int_0^1 \prod_{i=1}^t \phi_i(\theta) d\theta \leqslant \prod_{i=1}^t \Big(\int_0^1 |\phi_i(\theta)|^{p_i} \, d\theta \Big)^{1/p_i}$$

whenever $p_1, \dots, p_t > 1$ and $\frac{1}{p_1} + \cdots + \frac{1}{p_t} = 1$.

Question 8. Outline a complete proof of the Furstenberg–Sárközy theorem by assembling the above ingredients.

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