C3.10 Additive and Combinatorial Number Theory, Michaelmas 2020 Additional questions

Additional questions on the second half of the course for enthusiasts.

Question 1. Suppose that A is a bounded open subset of $\mathbb{R}^d$. Show that $\mathrm{vol}(A + A) \geqslant 2^d \mathrm{vol}(A)$.

Question 2. Construct a set $A \subset \{1, \dots, 10^5\}$ with no three elements in arithmetic progression, and with $|A| > 2012$.

Question 3. Give examples of each of the following:

(i) arbitrarily large sets of integers A, B such that $|A + B| \leqslant 1.01|A|$ and $|A - B| > |A|^{1.01}$;

(ii) arbitrarily large sets of integers A, B such that $|A + B| \leqslant 1.01|A|$ but $|A + 2B| > |A|^{1.01}$;

(iii) arbitrarily large sets of integers A, B such that $|A + B| > |A|^{1.01}$ but $|A + 2B| \leqslant 1.01|A + B|$;

Question 4. Let C be an arbitrary positive real number. Show that there is $\alpha > 0$ with the following property. For arbitrarily large values of N, there is a set $A \subset [N]$ with $|A| \geqslant \alpha N$, but containing fewer than $\alpha^C N^2$ three-term arithmetic progressions.

Question 5. Let $A \subset \mathbb{R}^d$ be a finite set which is symmetric (that is, $-x \in A$ if $x \in A$), contains 0, and is not contained in any proper subspace of $\mathbb{R}^d$.

(i) Explain why there is a nested sequence of subspaces $0 < V_1 < V_2 < \cdots < V_{d-1} < V_d = \mathbb{R}^d$ such that $A \cap V_{i+1} \neq A \cap V_i$ for all $i < d$.

(ii) Write $A_i := 2A \cap V_i$. Show that $2A_{i+1} \subsetneq A_{i+1} + V_i$.

(iii) For each i, choose $h \in 2A_{i+1} \setminus (A_{i+1} + V_i)$. Show that the sets $A + h_1, \dots, A + h_d$ are disjoint and conclude that $|5A| \geqslant d|A|$.

Question 6. By considering sets of the form

$$A = [N]^3 \cup \{(n, 0, 0), (0, n, 0), (0, 0, n) : n \leqslant LN\},$$

for appropriate L, show that for all K there are arbitrarily large sets $A \subset \mathbb{Z}^3$ with $|2A| \leqslant K|A|$ and $|3A| \geqslant \frac{1}{100} K^3 |A|$.

Question 7. Suppose that $A \subset \mathbb{Z}/N\mathbb{Z}$ is an arithmetic progression. Show that

$$\sum_{r \in \mathbb{Z}/N\mathbb{Z}} |\hat{1}_A(r)| \leqslant C \log N,$$

where C is an absolute constant.

Question 8. A set A in some abelian group is said to be a *Sidon set* if the only solutions to the equation $x + y = z + w$ with $x, y, z, w \in A$ are the trivial solutions in which $\{w, z\} = \{x, y\}$. Show that two Sidon sets of the same size are 2-isomorphic. Show that the set $\{(x, x^2) : x \in \mathbb{Z}/p\mathbb{Z}\}$ is a Sidon set in $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, and deduce that for any N there is a Sidon subset of $\{1, \dots, N\}$ of size at least $c\sqrt{N}$, for some absolute constant $c > 0$.

Question 9. Let A be a finite subset of $\mathbb{R}^n$, and let $s \geqslant 2$ be an integer. Show that A is Freiman s-isomorphic to a subset of $\mathbb{Z}$.

Question 10. Suppose that $A \subset \mathbb{Z}/N\mathbb{Z}$ is a set of size $\lfloor N/2 \rfloor$, and that $|\hat{1}_A(r)| \leqslant N^{-c}$ whenever $r \neq 0$, where c is some absolute constant. Show that if $N > N_0(c)$ is large enough then A intersects every arithmetic progression P in $\mathbb{Z}/N\mathbb{Z}$ of length at least $N/100$.

Question 11. Show that every set $A \subset \mathbb{Z}$ of size n contains a Sidon set of size at least $c\sqrt{n}$.

Question 12. Let p be a large prime, and suppose that $A \subset \mathbb{Z}/p\mathbb{Z}$ is a set of size at most $100 \log p$. Show that A is Freiman 2-isomorphic to a set of integers.

Question 13. Given a finite set $A \subset \mathbb{Z}$, define $\dim_s(A)$ to be the dimension of the space of Frieman s-homomorphisms from A to $\mathbb{Q}$, considered as a vector space over $\mathbb{Q}$. Show that if A is a random subset of $[n]$ (choosing each element independently at random with probability 1/2) then with probability tending to 1 as $n \to \infty$ we have $\dim_s(A) = 2$, for each fixed s.

Question 14. Suppose that N is a prime, and let $f : \mathbb{Z}/N\mathbb{Z} \to \{-1, 1\}$ be a function.

(i) Show that there is at least one value of r such that the discrete Fourier coefficient $\hat{f}(r)$ has $|\hat{f}(r)| \geqslant N^{-1/2}$.

(ii) Show that if $f(x) = (x|N)$, the Legendre symbol, then $|\hat{f}(r)| = N^{-1/2}$ for all r.

(iii) Deduce that the same is true if $f(x) = \pm(x + a|n)$, for any fixed $A \in \mathbb{Z}/N\mathbb{Z}$ and for either choice of sign $\pm$.

(iv) *Prove the converse: that is, if $|\hat{f}(r)| = N^{-1/2}$ for all r, then f has the form given in (iii).

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