

C3.8 Analytic Number Theory

Sheet 1 — MT20

Asymptotic notation and partial summation

1. Prove the following.

(a) $\log^4 X < X^{1/10}$ for all sufficiently large X ;

(b) $e^{\sqrt{\log X}} = O_\varepsilon(X^\varepsilon)$ for all $\varepsilon > 0$ and $X \geq 1$;

(c) $X(1 + e^{-\sqrt{\log X}}) + X^{3/4} \sin X \sim X$.

2. Let $\text{Li}(x) := \int_2^x \frac{dt}{\log t}$.

(a) Show that $\text{Li}(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$ for $x \geq 3$.

(b) Show that for any $k \geq 1$, $\text{Li}(x) = \sum_{j=1}^k \frac{(j-1)!x}{(\log x)^j} + O_k\left(\frac{k!^2 x}{(\log x)^{k+1}}\right)$ for $x \geq 3$.

3. In the following exercise, $a(X), b(X) \geq 2$ are functions tending to ∞ as $X \rightarrow \infty$. For each statement below, either give a proof of its correctness or a counterexample.

(a) If $a(X) \sim b(X)$ then $\frac{a(X)}{\log(a(X))} \sim \frac{b(X)}{\log(b(X))}$.

(b) If $a(X) - b(X) \rightarrow 0$ then $a(X) \sim b(X)$.

(c) If $a(X) \sim b(X)$ then $a(X) - b(X) \rightarrow 0$.

(d) If $a(X) \sim b(X)$ and $a'(X) := \sum_{y \leq X} a(y)$, $b'(X) := \sum_{y \leq X} b(y)$ then $a'(X) \sim b'(X)$.

(e) If $a'(X) \sim b'(X)$ where $a'(X) := \sum_{y \leq X} a(y)$, $b'(X) := \sum_{y \leq X} b(y)$, then $a(X) \sim b(X)$.

4. Show that there are arbitrarily large gaps between consecutive primes by

(a) using the bound $\pi(x) = O(x/\log x)$;

(b) considering the numbers $n! + 2, \dots, n! + n$.

Which of the two approaches gives the better bound?

5. Assume that $\pi(x) \sim x/\log x$.

(a) Show that $p_n \sim n \log n$, where p_n denotes the n th prime.

(b) Deduce that $p_{n+1} \sim p_n$ and $\sup_{p_n \leq x} (p_{n+1} - p_n) = o(x)$.

6. Let X be an integer.

(a) Show that

$$\log(X!) = \sum_{n \leq X} \log(n),$$

and

$$\log(X!) = \sum_{p \leq X} \log p \left(\left\lfloor \frac{X}{p} \right\rfloor + \left\lfloor \frac{X}{p^2} \right\rfloor + \dots \right).$$

(b) Show that

$$\sum_{n \leq X} \log n = X \log X - X + O(\log X),$$

and so

$$\sum_{p \leq X} \log p \left(\left\lfloor \frac{X}{p} \right\rfloor + \left\lfloor \frac{X}{p^2} \right\rfloor + \dots \right) = X \log X - X + O(\log X).$$

(c) Show that the contribution from the terms $\left\lfloor \frac{X}{p^k} \right\rfloor$ with $k \geq 2$ is $O(X)$.

(d) Deduce Mertens' first estimate

$$\sum_{p \leq X} \frac{\log p}{p} = \log X + O(1).$$

Explain why this remains valid even if X is not necessarily an integer.

7. Using Mertens' first estimate above, prove the second Mertens estimate: we have

$$\sum_{p \leq X} \frac{1}{p} = \log \log X + O(1).$$

Deduce that there are constants $c_1, c_2 > 0$ such that

$$\frac{c_1}{\log X} \leq \prod_{p \leq X} \left(1 - \frac{1}{p}\right) \leq \frac{c_2}{\log X}.$$

8. Let p_n denote the n th prime.

(a) Is it the case that, for sufficiently large n , the sequence $p_{n+1} - p_n$ is strictly increasing?

(b) Is it the case that, for sufficiently large n , the sequence $p_{n+1} - p_n$ is nondecreasing?