C3.8 Analytic Number Theory

Sheet 2 — MT20

Multiplicative functions

1. Give a simple description of the function $\phi \star 1$.

2. (a) Recall $\phi(n) = \#\{d \le n : \gcd(d,n) = 1\}$. Show that $\sum_n \phi(n) n^{-s} = \frac{\zeta(s-1)}{\zeta(s)}$ for $\Re s > 2$.

   (b) Let $\sigma(n) = \sum_{d|n} d$. Show that $\sum_n \sigma(n) n^{-s} = \zeta(s)\zeta(s-1)$ for $\Re s > 2$.

   (c) Let $\lambda(n)$ be the Liouville function, the completely multiplicative function equal to -1 on the primes. Show that $\sum_n \lambda(n) n^{-s} = \frac{\zeta(2s)}{\zeta(s)}$ for $\Re s > 1$.

   (d) Let $\tau_k(n) = \sum_{d_1 \dots d_k = n} 1$ be the number of ways of writing n as a product of k integers. Show that $\sum_n \tau_k(n) n^{-s} = \zeta(s)^k$ for $\Re s > 1$.

3. True or false: there is a constant C such that $\tau(n) \leqslant \log^C n$ for all sufficiently large n. (Do not assume the prime number theorem.)

4. (a) Show that for all $Y \ge 1$

$$\frac{1}{Y}\sum_{n\ge 1} \Lambda(n) \lfloor \frac{Y}{n} \rfloor = \frac{1}{Y}\sum_{n \leqslant Y} \log n.$$

   (b) Show that

$$\sum_{n \le Y} \log n = Y \log Y + O(Y).$$

   (c) Use part (a) and (b) to show that

$$\sum_{X/2 < n \leqslant X} \Lambda(n) \ll X.$$

   (*Hint:* $\lfloor 2x/d \rfloor - 2\lfloor x/d \rfloor \ge 0$, and is 1 when $x < d \le 2x$.)

5. Let $\Lambda_k(n) = (\mu \star \log^k)(n) = \sum_{de=n} \mu(d)(\log e)^k$. Show that

   (a) $-\mu(n)\log n = \sum_{de=n} \mu(d)\Lambda(e)$.

   (b) $\Lambda_{k+1}(n) = \Lambda_k(n)\log n + (\Lambda_k \star \Lambda)(n)$.

   (c) $\Lambda_k(n) = 0$ unless n has at most k distinct prime factors.

   (d) If $n = p_1 \dots p_k$ (for distinct primes $p_1, \dots, p_k$) then $\Lambda_k(n) = k!(\log p_1)\dots(\log p_k)$.

   (e) $0 \le \Lambda_k(n) \le (\log n)^k$.

   (*Hint: Use (b) in subsequent parts!*)

James Maynard: james.maynard@maths.ox.ac.uk

6. Give an asymptotic for $\sum_{n\leqslant X}\phi(n)$. (*Hint:* Using your answer to Question 1, or otherwise, first establish that the expression to be estimated is $\sum_{d\leqslant X}\mu(d)\sum_{m\leqslant X/d}m$.)

7. (a) Show that
$$\tau(n)=\sum_{\substack{d|n\\ d\leq n^{1/2}}}1+\sum_{\substack{d|n\\ d<n^{1/2}}}1.$$
   (b) Deduce that
$$\sum_{n\leq x}\tau(n)=2\sum_{d\leq x^{1/2}}\frac{x}{d}-x+O(x^{1/2}).$$
   (c) Let γ be the constant
$$\gamma=\int_1^\infty\frac{\{t\}}{t(t-\{t\})}dt.$$
   (Recall $\{t\}=t-\lfloor t\rfloor$ is the fractional part of t.) Show that
$$\gamma=\sum_{1\leq d\leq x}\frac{1}{d}-\log x+O(1/x),$$
   and deduce
$$\sum_{n\leq x}\tau(n)=x\log x+(2\gamma-1)x+O(x^{1/2}).$$

8. The aim of this question is to show that the prime number theorem implies that $M(X)=o(X)$, where $M(X):=\sum_{n\leqslant X}\mu(n)$.
   (a) Prove that if $n\neq 1$ then
$$-\mu(n)\log n=\sum_{ab=n}\mu(a)(\Lambda(b)-1).$$
   (b) Deduce that
$$|M(X)|\leqslant\frac{1}{\log X}\sum_a|\sum_{b\leqslant X/a}(\Lambda(b)-1)|+o(X).$$
   (c) Assuming the prime number theorem, show that indeed $M(X)=o(X)$.

James Maynard: james.maynard@maths.ox.ac.uk