C3.8 Analytic Number Theory, Michaelmas 2020

Past Exam Comments

The course has changed a bit from previous versions, so I have been asked about the relevance of past papers for revision purposes.

I have recently updated the specimen solutions for the course.

The Oxford Maths past paper archive `https://www.maths.ox.ac.uk/node/12897/41` has past papers from 2007- with the exception of 2013. In some of the earlier years there were additional questions on elliptic curves which are not relevant.

- **2007**: All questions look fine.
- **2008**: All questions look fine.
- **2009**: Ignore Question 1. For Question 3, instead assume for any choice of T there is the bound

  $$\frac{1}{\zeta(s)} = O(\log^7(2+|t|))$$

  in the region $s = \sigma + it$ with $\sigma \geq \alpha_T$ and $|t| \leq T$.
- **2010**: Ignore the final part of question 3 (using the Tauberian argument)
- **2011**: All questions look fine.
- **2012**: All questions look fine.
- **2014**: Ignore part 3(a), and assume for the rest of the question that if $\eta_1(x) \sim x^2/2$ then $\eta(x) \sim x$.
- **2015**: question , part (b): assume that the bounds for ζ hold for $\sigma \geq \alpha$, not just $\sigma \geq 1$.
- **2016**: In question 1, reinterpret the phrase 'has abscissa of convergence σ_0' with 'converges absolutely for $\Re(s) > \sigma_0$'. Ignore 3(c).
- **2017**: In quesion 1, $D_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$. In question 3(a), reinterpret the question as 'state the explicit formula'. For 3(c), reinterpret the question as showing that

  $$\sum_{n<x} \Lambda(n) = x + O(x^{\alpha+o(1)})$$

  as $x \to \infty$.
- **2018**: Ignore question 2. In 3(d), reinterpret the question as showing that

  $$\sum_{n<x} \tau(n)^2 = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\zeta(s)^4}{\zeta(2s)} \frac{x^s}{s} ds$$

In 3(e), reinterpret the question as showing the residue of $\zeta(s)^4x^s/(s\zeta(2s))$ is $cx\log^3 x + O(x\log^2 x)$ for some constant $c > 0$ to be specified. In 3(f), instead explain briefly why one expects that

$$\sum_{n<x}\tau(n)^2 \sim cx\log^3 x.$$

- **2019**: All questions look fine.
- **2020**: All questions look fine.

If you wish to look for more example questions, then I would reccommend looking in the book 'Multiplicative Number Theory I: Classical Theory' by Montgomery and Vaughan.

Together with the examples sheets, these give a good bank of questions based on the ideas of all chapters, except possibly chapters 6,7 and 12. Chapters 6 and 7 are more auxilliary chapters, and so to acknowledge this lack of examples any exam questions will stick closely to the content covered in lectures.

Here are two further example questions based on Chapter 12:

Question 1 (Primes in short intervals under the Riemann Hypothesis)**.** Assume the Riemann Hypothesis holds. Let $2 \le x$ and $1 \ge \delta \ge (\log x)^{10}/x$. Show that for all but $o(x)$ values of $t \le x$ we have

$$\#\{p \in [t, t+\delta t]\} = \frac{\delta t}{\log t} + o\Big(\frac{\delta t}{\log t}\Big).$$

Question 2 (Improved error term)**.** Using Fact 12.1 (Improved zero free region), prove an improved Prime Number Theorem of the form

$$\psi(x) = x + O\Big(x\exp(-c(\log x)^{3/5-\epsilon}).$$

Recap

To recap, the key content of the course is the following:

- Using partial summation to express sums of one arithmetic function in terms of sums of related arithmetic functions.
- Basic asymptotic bounds and manipulations of arithmetic functions.
- Convergence of Euler products and Dirichlet series of arithmetic functions, and interactions with multiplicative functions.
- Using Perrons formula to express the sum of an arithmetic function as an integral of a Dirichlet series.
- Using Cauchy's residue theorem and basic facts about a Dirichlet series to move the integration and estimate a contour integral coming from perrons formula.

- Proof of the functional equation for $\zeta(s)$.
- Proof of the partial fraction expansion for $\zeta(s)$.
- Proof of the zero free region for $\zeta(s)$.
- Proof of the explicit formula.
- Proof of the prime number theorem.
- Applications of the prime number theorem.

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