C3.4 ALGEBRAIC GEOMETRY 2020 - WARM-UP EXERCISE SHEET - NOT TO BE HANDED IN†

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Exercise 1. Varieties: solution sets of polynomials.

Let V_0, V_1, V_2 be the solution sets respectively of the three equations

$$y^2 = x^3 \qquad y^2 = x^3 + x \qquad y^2 = x^3 + x^2.$$

Draw in $\mathbb{R}^2$ the solutions, and check that V_0 has a cusp at 0, V_1 has a vertical tangent at 0, and V_2 self-intersects itself at 0. Now work in $\mathbb{C}^2$, what complex solutions are missing?[1]

Exercise 2. Blow-ups.

The *blow-up* of V_2 at $(0,0)$ is defined as the solution set[2]

$$\widetilde{V_2} = \{((x,y),[z_0,z_1]) \in \mathbb{C}^2 \times \mathbb{CP}^1 : y^2 - x^3 - x^2 = 0, xz_1 = yz_0\}.$$

Intuitively, the $\mathbb{CP}^1$ keeps track of the slope $y/x = z_1/z_0$. Show that projection $\widetilde{V_2} \to V_2 \subset \mathbb{C}^2$ to the first factor is a bijection except over $(0,0)$. Does the curve $\widetilde{V_2}$ self-intersect?

Exercise 3. $\mathbb{C}$-algebras.

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials over $\mathbb{C}$ in n variables. Show that a homomorphism $\varphi : R \to S$ of $\mathbb{C}$-algebras[3] is completely determined by the choice of n elements in S, namely the images under φ of $x_1, \ldots, x_n$. Show that S is a finitely generated[4] $\mathbb{C}$-algebra if and only if there is a surjective such $\varphi : R \to S$, for some n. Construct an isomorphism

$$S \cong \mathbb{C}[x_1, \ldots, x_n]/I \quad \text{for some ideal } I \subset \mathbb{C}[x_1, \ldots, x_n].$$

Is this isomorphism unique? (if not, construct a counterexample).

Exercise 4. The functions on a variety.

Consider one of the curves V from Exercise 1, defined by the relevant equation $f = 0$.

Let $\mathrm{Hom}(V, \mathbb{C})$ be the set of all complex functions $V \to \mathbb{C}$ which can be expressed as polynomials over $\mathbb{C}$ in x, y. Check that $\mathrm{Hom}(V, \mathbb{C})$ is a $\mathbb{C}$-algebra.

Consider the $\mathbb{C}$-algebra $\mathbb{C}[V]$, called *coordinate ring of* V, defined by quotienting $\mathbb{C}[x,y]$ by the ideal generated by f,

$$\mathbb{C}[V] = \mathbb{C}[x,y]/(f).$$

Explain why this $\mathbb{C}$-algebra is isomorphic to $\mathrm{Hom}(V, \mathbb{C})$.

The fraction field $\mathbb{C}(V) = \mathrm{Frac}\,\mathbb{C}[V]$ is a field extension of $\mathbb{C}$, and the *dimension* of V is the *transcendence degree* of this extension.[5] Show that our curves V have dimension 1.

Exercise 5. Tangent spaces.

Let V be one of the curves in Exercise 1 defined by the relevant polynomial f. Let $p \in V$. Consider a (complex) line $\ell(t) = p + tv$ through p, parametrized by $t \in \mathbb{C}$, with velocity $v \in \mathbb{C}^2$. The line ℓ is *tangent* to V at p if the polynomial $f(\ell(t))$ in t has a zero of order at least two at $t = 0$. What are the lines tangent to V_0, V_1, V_2?

The *tangent space* T_pV at $p \in V$ is the union of all lines tangent to V at p. Convince yourself that T_pV is a vector space. Say that p is a *singular point* if the vector space dimension $\dim_k T_pV$ of T_pV does not equal $\dim V$ (in our case, $\dim V = 1$). Find the singular points of V_0, V_1, V_2.

Show that by doing a blow-up of V_0 you obtain a curve without singularities.

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† The aim of this sheet is to flag up some themes from the course without detailed technical machinery. you should be able to do these questions using knowledge of earlier courses.

[1] Hint: how many solutions do you expect if you intersect the curve with $x = c$, some constant?

[2] Recall that the *complex projective line* is $\mathbb{CP}^1 = (\mathbb{C}^2 \setminus (0,0)) / \sim$ where we identify $(z_0, z_1) \sim (\lambda z_0, \lambda z_1)$ for any $\lambda \in \mathbb{C} \setminus 0$, and we typically denote the equivalence class by $[z_0 : z_1]$. Notice this space is covered by two open sets: $z_0 \neq 0$ and $z_1 \neq 0$. If $z_0 \neq 0$, we can rescale so that $[z_0 : z_1] = [1 : z]$, so that open set is just a copy of $\mathbb{C}$ parametrized by the variable $z = z_1/z_0$. Similarly $z_1 \neq 0$ is a copy of $\mathbb{C}$ parametrized by $w = z_0/z_1$. The overlap of the two open sets is a copy of $\mathbb{C} \setminus 0$, and there the two parameters are related by $z = 1/w$.

[3] A $\mathbb{C}$-algebra is a ring which is also a vector space over $\mathbb{C}$, satisfying the obvious axioms. A homomorphism of $\mathbb{C}$-algebras is a ring hom (in particular 1 maps to 1) which is also a linear hom of vector spaces over $\mathbb{C}$.

[4] A $\mathbb{C}$-algebra is *finitely generated* by $a_1, \ldots, a_n$ if every element is a polynomial over $\mathbb{C}$ in the $a_1, \ldots, a_n$.

[5] i.e. the maximal number of algebraically independent variables of $\mathbb{C}(V)$ over $\mathbb{C}$. Fact: for an extension $\mathbb{C} \hookrightarrow K$, if $y_1, \ldots, y_m \in K$ are algebraically independent over $\mathbb{C}$, and $\mathbb{C}(y_1, \ldots, y_m) = \mathrm{Frac}\,\mathbb{C}[y_1, \ldots, y_m] \subset K$ is an algebraic extension, then m is the transcendence degree $\mathrm{trdeg}_{\mathbb{C}} K$.