C3.4 ALGEBRAIC GEOMETRY 2020 - EXERCISE SHEET 1

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In these questions, k denotes an algebraically closed field.

(1) **Zariski topology**
  (a) Verify that arbitrary intersections and finite unions of affine varieties are affine varieties.
  (b) List the open and closed subsets of $\mathbb{A}^1_k$ in the Zariski topology.
  (c) Describe carefully the Zariski closed subsets of $\mathbb{A}^2_k$, proving your statements.
  (d) Show that the Zariski topology on $\mathbb{A}^2_k$ is *not* the product topology on $\mathbb{A}^1_k \times \mathbb{A}^1_k$.

(2) **Irreducibility**
  (a) Show that affine n-space $\mathbb{A}^n_k$ is irreducible.
  (b) Show that an affine variety $X \subset \mathbb{A}^n_k$ is irreducible if and only if every non-empty open subset $U \subset X$ is dense in the Zariski topology[1].
  (c) Let X be an irreducible affine variety. Show that any two non-empty open sets intersect in a non-empty open dense set.

(3) **The variety of nilpotent matrices** We work in the affine space $\mathbb{A}^4$ parametrising 2×2 matrices over k, with variables being the matrix entries x_{ij}.
  (a) Prove that the following conditions are equivalent for a 2×2 matrix A over a field k:
    - A is *nilpotent*: there exists an $n \geq 1$ such that $A^n = 0$;
    - $A^2 = 0$;
    - $\det A = \mathrm{tr}\ A = 0$.

  Let $I \lhd R = k[x_{11}, x_{12}, x_{21}, x_{22}]$ be the ideal formed by the polynomials $d = \det A, t = \mathrm{tr}\ A$, viewed as polynomials in the matrix entries. Let $J \lhd R$ be the ideal formed by the entries of A^2, as polynomials in the matrix entries. Show the following.
  (b) The ideal J is not radical: it contains a power of t but not t itself.
  (c) The ideal I is radical[2].
  (d) Deduce that $X = \mathbb{V}(I) = \mathbb{V}(J) \subset \mathbb{A}^4$ with $\sqrt{J} = I$, and conversely $\mathbb{I}(X) = I$.

(4) **Reduced algebras as coordinate rings**
  (a) Show that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$ for ideals I, J of a finitely generated k-algebra R.
  (b) Show that the ideal $(xy, xz) \subset k[x, y, z]$ is radical but not prime. Sketch the variety it defines in $\mathbb{A}^3_k$.
  (c) Let $X \subset \mathbb{A}^n_k$ be an affine variety. Show that a radical ideal in $k[X]$ is the intersection of all the maximal ideals containing it[3].
  (d) *(Harder)* Show that a variety $X \subset \mathbb{A}^n_k$ is a union of two disjoint closed subvarieties if and only if its coordinate ring $k[X]$ may be written as the product of two non-trivial finitely generated reduced k-algebras[4].

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[1] Hint: the converse statement is easier to show.

[2] Hint: aim to show that I is prime and therefore radical. Show this by mapping R/I to an isomorphic ring using the linear generator.

[3] Hint: using methods of this course, it is easier to first translate this into a geometrical statement, and prove that. For an algebraic proof, you might find helpful the following theorem due to Krull: the nilradical $\mathrm{nil}(A) = \{x : x^m = 0 \text{ some } m\}$ of a ring A equals the intersection of all its prime ideals.

[4] Hint: recall the algebraic form of the Chinese Remainder Theorem: if I_1, I_2 are coprime ideals in a ring R, meaning $I_1 + I_2 = R$, then $I_1 \cap I_2 = I_1 \cdot I_2$ and there is a ring isomorphism $R/(I_1 \cap I_2) \to R/I_1 \times R/I_2$ given by $f \mapsto (f + I_1, f + I_2)$.

(5) **The pull-back map between coordinate rings.** Suppose that $F : X \to Y$ is a morphism of affine varieties over a field k, associated to a map $F^* : k[Y] \to k[X]$ between their coordinate rings.
   (a) Show that F^* is injective if and only if F is dominant, i.e. the image set $F(X)$ is dense in Y.
   (b) Show that F^* is surjective if and only if F defines an isomorphism between X and some algebraic subvariety of Y.
   (c) Find an example where F is injective but F^* is not surjective.

(6) **The affine normal curve.** Consider the homomorphism of rings

$$F^* : k[x_0, \ldots, x_{n-1}] \to k[t]$$

given by $x_i \mapsto t^i$.
   (a) Show that the corresponding morphism of affine varieties $F : \mathbb{A}^1_k \to \mathbb{A}^n_k$ defines an isomorphism between $\mathbb{A}^1_k$ and its image under F.
   (b) Find generators for the ideal defining the image of F in $\mathbb{A}^n_k$.