C3.4 ALGEBRAIC GEOMETRY 2020 - EXERCISE SHEET 2

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(1) **Projective closures and affine cones**

(a) Let X be the parabola $\mathbb{V}(y-x^2) \subset \mathbb{A}^2$. What is its projective closure $\bar{X} \subset \mathbb{P}^2$? Draw the affine cone $\hat{\bar{X}}$ over $\bar{X}$, in $\mathbb{A}^3$, and identify the line corresponding to the "point at infinity" on $\bar{X}$.

(b) Show that the affine varieties $\mathbb{V}(y-x^2) \subset \mathbb{A}^2$ and $\mathbb{V}(y-x^3) \subset \mathbb{A}^2$ are isomorphic. Recalling that $z^2 = x^3$ is a cuspidal cubic with a singularity at zero, can you give an intuitive explanation[1] why their two projective closures in $\mathbb{P}^2$ are *not* isomorphic?

(2) **The Twisted Cubic.** The projective variety $C = \mathbb{V}(F_0, F_1, F_2) \subset \mathbb{P}^3$, where

$$\begin{aligned} F_0(z_0, z_1, z_2, z_3) &= z_0 z_2 - z_1^2 \\ F_1(z_0, z_1, z_2, z_3) &= z_0 z_3 - z_1 z_2 \\ F_2(z_0, z_1, z_2, z_3) &= z_1 z_3 - z_2^2, \end{aligned}$$

is known as the *twisted cubic*.

(a) Show that C is equal to the image of the Veronese map

$$\begin{aligned} \nu : \mathbb{P}^1 &\to \mathbb{P}^3 \\ \nu : [x_0 : x_1] &\mapsto [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3]. \end{aligned}$$

(b) Restrict to the affine patch $U_0 \subset \mathbb{P}^3$ given by setting $z_0 = 1$. Show that $C \cap U_0$ is equal to $\mathbb{V}(f_0, f_1) \subset \mathbb{A}^3$, where $f_i(z_1, z_2, z_3) := F_i(1, z_1, z_2, z_3)$ for $i = 1, 2$.

(c) For $i = 0, 1, 2$ we write Q_i for the quadric hypersurface $\mathbb{V}(F_i) \subset \mathbb{P}^3$. Show that, for $i \neq j$, the hypersurfaces Q_i and Q_j intersect in the union of C and a line L. Therefore no two of them alone may be used to define C. Deduce that the homogenizations of the generators of an affine ideal do not necessarily generate the homogeneous ideal of the projective closure, showing that indeed we need to homogenise *all* elements of the affine ideal.

Cultural Remark: *The codimension of C is 2 in $\mathbb{P}^3$ (it is a curve), but it can be proved that its ideal cannot be generated by 2 polynomials (we have seen that no two of F_0, F_1, F_2 generate), so C is not a* complete intersection. *But $C \cap U_i$ is a complete intersection, as we have just seen.*

(3) **Veronese varieties**

(a) Show that any projective variety is isomorphic to the intersection of a Veronese variety with a linear space, the projectivisation $\mathbb{P}(V)$ of some k-vector subspace $V \subset k^{n+1}$.

(b) Deduce that any projective variety is isomorphic to an intersection of quadric hypersurfaces.

[1] We do not have the tools yet to *prove* that these are non-isomorphic, but you should be able to "see" that this is true.

(4) **The ruled surface.** The image of the Segre morphism $\sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) = \Sigma_{1,1} \subset \mathbb{P}^3$ is known as the *ruled surface*.
  (a) What equations define $\Sigma_{1,1}$ as a subvariety of $\mathbb{P}^3$?
  (b) What are the images in $\Sigma_{1,1}$ of $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$? Show that through any point in $\Sigma_{1,1}$ there are two lines lying in $\Sigma_{1,1}$.
  (c) Exhibit some disjoint lines in $\Sigma_{1,1}$. Recall that $\mathbb{P}^1 \times \mathbb{P}^1 \cong \Sigma_{1,1}$. Is this isomorphic to $\mathbb{P}^2$? Draw the "real cartoons" of either surface.

(5) **Rational normal curves**
  (a) Let $G(x_0, x_1) = \Pi_{i=1}^{d+1}(b_i x_0 - a_i x_1)$ be a homogeneous degree $(d+1)$ polynomial with distinct roots $[a_i : b_i] \in \mathbb{P}^1$. Show that $H_i(x_0, x_1) = G(x_0, x_1)/(b_i x_0 - a_i x_1)$ form a basis for the space of homogeneous polynomials of degree d.
  (b) Deduce that the image of the map $\mu_d \colon \mathbb{P}^1 \to \mathbb{P}^d$ defined by

$$[x_0 : x_1] \mapsto [H_1(x_0, x_1) : \cdots : H_{d+1}(x_0, x_1)]$$

  is projectively equivalent to the image of the Veronese embedding, that is, it is a rational normal curve.
  (c) What is the image of the point $[a_i : b_i]$? If (a_i, b_i) are nonzero for all i, what is the image of $[1 : 0]$ and $[0 : 1]$?
  (d) Deduce that through any $d+3$ points in general position[2] in $\mathbb{P}^d$, there passes a unique rational normal curve.

(6) **Projective variety corresponding to a graded ring.** If $R = \sum_{d \geq 0} R_d$ is a graded ring and $e \geq 1$ is an integer, we define

$$R^{(e)} := \sum_{d \geq 0} R_{de}.$$

We define a grading on $R^{(e)}$ by letting $R_d^{(e)} := R_{de}$.
  (a) Find $k[x_0, x_1]^{(2)}$, expressing it in the form $k[z_0, \ldots, z_n]/I$ for some n and I.
  (b) Find the homogeneous coordinate rings $S(\mathbb{P}^1)$ and[3] $S(\nu_2(\mathbb{P}^1))$. Comment in the context of part (a).
  (c) More generally, show that $S(\nu_e(\mathbb{P}^n)) \cong k[x_0, \ldots, x_n]^{(e)}$, and hence that $k[x_0, \ldots, x_n]^{(e)}$ defines the same projective variety as $k[x_0, \ldots, x_n]$.
  (d) Are $k[x_0, \ldots, x_n]^{(e)}$ and $k[x_0, \ldots, x_n]$ isomorphic as graded k-algebras? Are they isomorphic as (ungraded) k-algebras? What does this imply about the affine cones of $\nu_e(\mathbb{P}^n)$ and $\mathbb{P}^n$?

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[2]Recall that this means that any subset of $d+1$ of these points do not lie on a hyperplane in $\mathbb{P}^d$.
[3]Recall that the Veronese morphism $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$ is an isomorphism onto its image.