C3.4 ALGEBRAIC GEOMETRY 2020 - EXERCISE SHEET 3

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(1) **Another set of equations for the twisted cubic curve**

(a) Let $Q = \{x_0x_2 = x_1^2\} \subset \mathbb{P}^3$ and $F = \{x_0x_3^2 - 2x_1x_2x_3 + x_2^3 = 0\} \subset \mathbb{P}^3$. Prove that $C = Q \cap F \subset \mathbb{P}^3$ is the twisted cubic curve, the image of the third Veronese embedding $\nu_3(\mathbb{P}^1) \subset \mathbb{P}^3$.
Hint: multiply the second equation with x_2 and use the first equation to put it in the form of a perfect square...

(b) Does this mean that the ideal $\mathbb{I}(\nu_3(\mathbb{P}^1))$ can in fact be generated by two elements? (Compare Sheet 2, Question (2).)

(2) **Projecting a space curve**
Let $Q_0 = \{x_0x_3 = x_1^2\} \subset \mathbb{P}^3$ and $Q_1 = \{x_1x_3 = x_2^2\} \subset \mathbb{P}^3$. Consider the projective variety $C = Q_0 \cap Q_1 \subset \mathbb{P}^3$. Show that the formula $\pi\colon [x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : x_1 : x_2]$ defined on a Zariski open subset of C can be extended to a projective morphism $\pi\colon C \to \mathbb{P}^2$. Show that π maps C isomorphically to the projective variety (plane curve) $D = \{x_0x_2^2 = x_1^3\} \subset \mathbb{P}^2$.
Hint: to extend π, try to express the ratios $x_0 : x_1 : x_2$ in a different way using the equations of C.

(3) **Degree of a conic, explicitly**
Let $C = \{x_0^2 + x_1^2 + x_2^2 = 0\} \subset \mathbb{P}^2_{x_0,x_1,x_2}$. Consider lines

$$L = \{a_0x_0 + a_1x_1 + a_2x_2 = 0\} \subset \mathbb{P}^2_{x_0,x_1,x_2}.$$

Find the set of coefficients (a_0, a_1, a_2) so that the intersection $L \cap C$ *does not* consist of two distinct points.
Cultural Remark: *The coefficients a_i parametrize the* dual projective plane $\mathbb{P}^2_{a_0,a_1,a_2}$, *which plays the role of the general Grassmannian in the definition of degree. The locus you have just computed is the bad locus inside this Grassmannian; its complement is the generic locus in the definition of degree.*

(4) **Dimension, degree and Hilbert polynomial**

(a) Show carefully that if X is a reducible projective variety with equidimensional irreducible components X_i, then $\deg X = \sum_i \deg X_i$.

(b) Find the Hilbert polynomial of the Veronese image $\nu_d(\mathbb{P}^n)$. Verify that this projective variety has dimension n and degree d^n.

(c) Compute the degree $\deg \nu_d(\mathbb{P}^1)$ directly, without using the Hilbert polynomial.

(d) Let F be a homogeneous irreducible polynomial of degree d in $k[x_0, \dots, x_n]$, and let $X = \mathbb{V}(F) \subset \mathbb{P}^n$. Find the Hilbert polynomial of X. Deduce that, as expected, $\dim X = n - 1$ and $\deg X = d$.

(5) **Affine and quasi-projective varieties**

(a) Find an open affine cover of $\mathbb{A}^2 \setminus \{(0, 0)\}$.

(b) Show that $\mathrm{GL}(n, k)$ is an affine variety, i.e. that it is isomorphic as a quasi-projective variety to a Zariski closed subset of an affine space.

(c) Let X be an affine variety and $f \in k[X]$. Show that f vanishes nowhere on X if and only if f is invertible in $k[X]$.