C3.4 ALGEBRAIC GEOMETRY 2020 - EXERCISE SHEET 4

Comments and corrections are welcome: szendroi@maths.ox.ac.uk

(1) **Tangent Spaces.**

(a) Show that if $\text{char}(k)$ does not divide d, then the hypersurface $\mathbb{V}(x_0^d + \ldots + x_n^d) \subset \mathbb{P}^n$ is nonsingular.

(b) Under the same assumption on k, find the singular locus of the hypersurface $\mathbb{V}(x_0^d + \ldots + x_{n-1}^d) \subset \mathbb{P}^n$.

(c) Let $X_1 = \mathbb{V}(xy(x-y)) \subset \mathbb{A}^2$ and $X_2 = \mathbb{V}(xy, yz, zx) \subset \mathbb{A}^3$. Decompose these varieties into irreducible components. By computing the dimension of tangent spaces at various points, show that X_1 and X_2 are not isomorphic.

(2) **Rational and birational maps.**

(a) Let $F : X \dashrightarrow Y$ be a rational map of quasi-projective varieties, with X irreducible. If (U, f) is a representation for F, with U affine, show that

$$\{(u, f(u)) : u \in U\} \subset U \times Y$$

is a closed subvariety. Define the *graph* Γ_F of F to be the (Zariski) closure of $\{(u, f(u)) : u \in U\} \subset X \times Y$. Show that the projection from the graph $\Gamma_F \to X$ is a birational equivalence.

(b) Define $F : \mathbb{A}^2 \to \mathbb{A}^1$ by $F(x, y) = \frac{y}{x}$. Find the equation defining $\Gamma_F \subset \mathbb{A}^3$.

(c) Find a birational equivalence $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.
Hint. Combine two linear projections; equivalently, find isomorphic open sets in the two varieties!

(3) **Resolution of singularities.** In these examples, attempt a resolution of singularities of the given variety by successively blowing up singular points. The last two examples are optional and might land you in heavier calculations.

(a) Find a resolution of singularities of the curve $C_1 = \mathbb{V}(y^2 - x^2 - x^3) \subset \mathbb{A}^2$. Deduce that C_1 is rational, in other words birationally equivalent to $\mathbb{P}^1$.
Hint. Try blowing up the curve at the origin, and consider the map that projects to the exceptional divisor.

(b) Desingularise the curve $C_2 = \mathbb{V}(y^2 - x^4 - x^5) \subset \mathbb{A}^2$. Draw a picture of the series of blow-ups.

(c) Desingularise the surface $S_2 = \mathbb{V}(-xy + z^2) \subset \mathbb{A}^3$.

(d) Find an inductive procedure to desingularise the surface $S_n = \mathbb{V}(-xy + z^n) \subset \mathbb{A}^3$ for a positive integer $n > 2$.

(e) (Optional, harder) Desingularise the surfaces

$$T_1 = \{x^2 + y^3 + z^3 = 0\} \subset \mathbb{A}^3$$

and

$$T_2 = \{x^2 + y^3 + z^4 = 0\} \subset \mathbb{A}^3.$$