C3.4 Algebraic Geometry

Lecture 1: Introduction

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What is algebraic geometry about?

Algebraic geometry is the study of geometric spaces given by polynomial equations. More precisely, it is the study of geometric spaces given by the **vanishing** of polynomial equations of (Cartesian) coordinates.

Polynomials: very natural notion. As soon as we have "numbers" that we can add and multiply, we can take a bunch of variables and write down polynomials.

We get polynomial rings $S[x_1, x_2, \ldots]$ over rings S.

In this course,

- we will consider polynomial rings over fields k, and
- we will have a finite number of indeterminates (variables), so

$$R = k[x_1, \ldots, x_n].$$

We will think of variables $x_1, \ldots, x_n$ as Cartesian coordinates on affine space

$$p = (x_1, \ldots, x_n) \in \mathbb{A}^n = \mathbb{A}^n_k = k^n.$$

Familiar examples

You already know some examples from earlier studies!

- Fix constants $a_1, \ldots, a_n, c \in k$. Then
$$H = \left\{ \sum_{i=1}^{n} a_i x_i - c = 0 \right\} \subset \mathbb{A}^n$$
is an **affine hyperplane**. If $c = 0$, then H is a **hyperplane** (codimension one linear subspace).
- More generally, if we take several such equations, we get **affine linear subspaces**, respectively (if all constants are zero) **linear subspaces**.
- Take $k = \mathbb{R}$. Then
$$\{x^2 + y^2 - 1 = 0\} \subset \mathbb{A}^2_{\mathbb{R}}$$
is a **circle**. More generally, any quadratic equation in (x, y) describes a (real) **plane conic** or **conic section**.

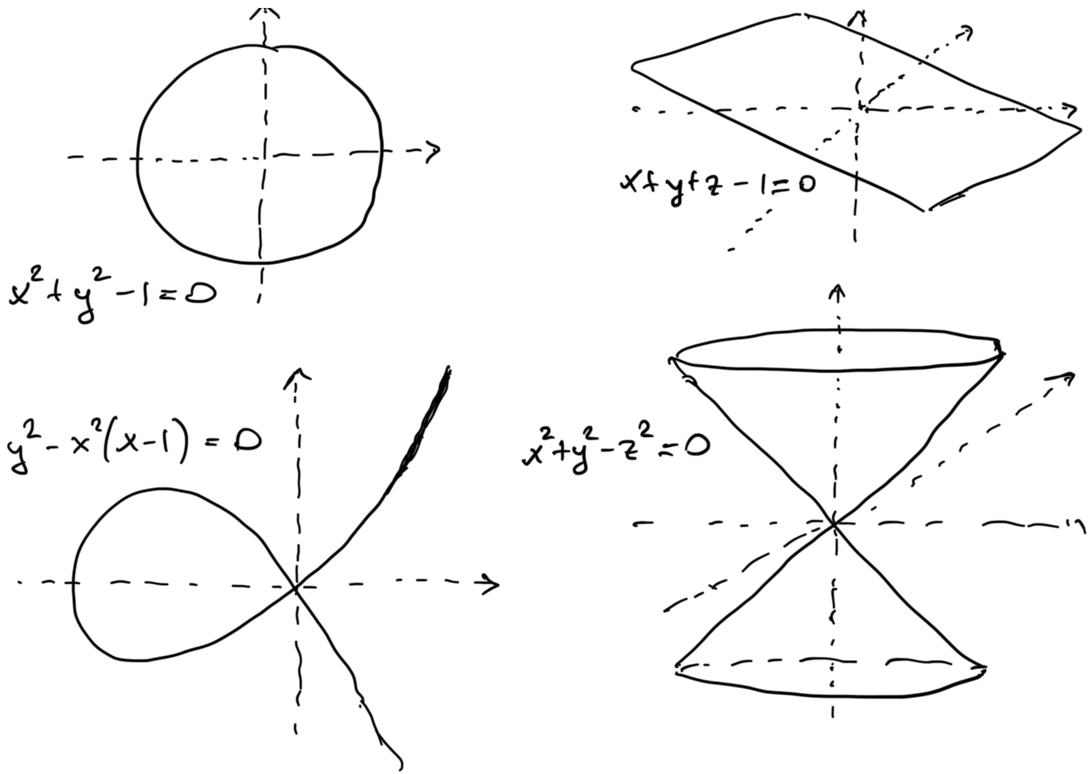
More familiar examples

- Take $k = \mathbb{C}$. Then
$$\{p(x, y) = 0\} \subset \mathbb{A}^2_{\mathbb{C}}$$
for any polynomial $p \in \mathbb{C}[x, y]$ is a (complex) **affine plane curve**, studied in the Oxford Part B course on Complex Algebraic Curves (and of course elsewhere).

- Take $k = \mathbb{R}$ again. Then
$$\{x^2 + y^2 - z^2 - c = 0\} \subset \mathbb{A}^3_{\mathbb{R}}$$
is a **hyperboloid**, with number of sheets depending on the sign of $c \neq 0$. For $c = 0$, we get the **quadric cone**.

- With $k = \mathbb{R}$ and arbitrary n, we have the (real) $(n-1)$**-sphere**
$$S^{n-1} = \left\{ \sum_{i=1}^{n} x_i^2 - 1 = 0 \right\} \subset \mathbb{A}^n_{\mathbb{R}}.$$

Familiar examples over $\mathbb{R}$ in pictures



General features

- A lot of the theory works for arbitrary fields k. We will assume
  - k has characteristic 0;
  - k is algebraically closed.
- Just take $k = \mathbb{C}$ if you wish!
- Number of variables will be arbitrary (finite!).
- Number of equations will also be arbitrary (finite! but now not a restriction).
- Drawing pictures remains a lot easier if $k = \mathbb{R}$ and $n \leq 3$...
- Don't forget other fields such as $\mathbb{F}_p, \mathbb{F}_q, \mathbb{Q}_p, \mathbb{C}(t), \ldots$ in further studies.

Applications of algebraic geometry

- Within pure mathematics: interacts with many different fields!
  - Key role in Wiles' proof of Fermat's Last Theorem
- Recent prominent role in theoretical physics
  - Spacetime models in string theory from algebraic geometry via supersymmetry
- Prominent applications in other areas
  - Algebraic robotics: describe motion of automate constrained by polynomial conditions.
  - Cryptography: cryptosystems from geometry (elliptic curves, abelian varieties...)
  - Algebraic systems biology: describe equilibria of complicated polynomial interaction systems

Sources of information

- Lecture notes by Prof Ritter on course website
  - Will follow the same notation.
  - The material in lectures forms a subset of the notes; will ignore categorical aspects but feel free to read those sections for a different, important point of view.
- Books
  - Many books around. Reid: UAG is perhaps the most useful. Hartshorne: Algebraic geometry is the “bible” but is too advanced just for this course.
- Problem sheets
  - There will be 5 problem sheets in total. Sheet 0 is not for handing in.

Commutative algebra in algebraic geometry

k denotes a field, algebraically closed and of characteristic 0, with unit $1 \in k$.

R a finitely generated, unital, commutative k-algebra: finitely generated as a commutative ring, has multiplication by elements of k, also has unit $1 \in R$. For example,

$$R = k[x_1, \ldots, x_n].$$

We will consider ideals $I \lhd R$, their intersections, products, quotient rings, etc. Also ring/algebra homomorphisms, kernels, images, etc.

I will quote results from Commutative Algebra. They can be taken without proof in this course; the Part B course Commutative algebra proves most of these results.

A simple but important proposition

Proposition Let k be a field, S a finitely generated commutative k-algebra. Then

$$S \cong k[x_1, \ldots, x_n]/I$$

for some n and an ideal $I \lhd k[x_1, \ldots, x_n]$.

Proof Let $s_1, \ldots s_n \in S$ be a set of k-algebra generators of S. Consider the ring homomorphism

$$\varphi \colon k[x_1, \ldots, x_n] \to S$$

defined by $\varphi(x_i) = s_i$. Then φ is surjective, since s_i generate S. Considering

$$I = \ker \varphi \lhd k[x_1, \ldots, x_n],$$

we get indeed

$$S \cong k[x_1, \ldots, x_n]/I$$

by the Isomorphism Theorem for rings. $\square$

Vanishing sets

We are working in the space $k^n = \{a = (a_1, \dots, a_n) : a_j \in k\}$.

This space corresponds to the polynomial ring $R = k[x_1, \dots, x_n]$.

$X \subset k^n$ is an **affine (algebraic) variety**, if $X = \mathbb{V}(I)$ for some ideal $I \subset R$, where

$$\mathbb{V}(I) = \{a \in k^n : f(a) = 0 \text{ for all } f \in I\} \subset k^n.$$

Examples

- For the zero ideal, $\mathbb{V}(0) = k^n$.
- For the ideal $\langle 1 \rangle = R$ generated by the identity, $\mathbb{V}(R) = \emptyset$.
- For some nonconstant $f \in R \setminus k$ generating principal ideal $\langle f \rangle \lhd R$, we get

$$V_f = \mathbb{V}(\langle f \rangle) = \{a \in k^n : f(a) = 0\},$$

  the **hypersurface** defined by f.

An easy but important example

Let $a = (a_1, \ldots, a_n) \in k^n$ and consider

$$\mathfrak{m}_a = \langle x_1 - a_1, \ldots, x_n - a_n \rangle \lhd R.$$

The following are all easy to check:

- $\mathbb{V}(\mathfrak{m}_a) = \{a\} \subset k^n$.
- The ideal $\mathfrak{m}_a \lhd R$ is the kernel of the evaluation homomorphism

  $$\mathrm{ev_a} \colon R \to k$$

  defined by $f \mapsto f(a)$.
- The ideal $\mathfrak{m}_a \lhd R$ is a maximal ideal of R.

Recall that an ideal $\mathfrak{m} \lhd R$ of a ring is **maximal** if it is not equal to R, nor is properly contained in another proper ideal of R. Remember that $\mathfrak{m} \lhd R$ is maximal if and only if the quotient $R/\mathfrak{m}$ is a field.

Some basic properties of vanishing sets

1. $I \subset J \Rightarrow \mathbb{V}(I) \supset \mathbb{V}(J)$.
2. $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cdot J) = \mathbb{V}(I \cap J)$.
3. $\mathbb{V}(I) \cap \mathbb{V}(J) = \mathbb{V}(I + J)$. (Note: $\langle I \cup J \rangle = I + J$.)
4. $\mathbb{V}(I), \mathbb{V}(J)$ are disjoint if and only if I, J are relatively prime (i.e. $I + J = \langle 1 \rangle$)

The proofs are easy exercises.

Hilbert's Basis Theorem

Hilbert's Basis Theorem $R = k[x_1, \ldots, x_n]$ is a Noetherian ring. In other words, it satisfies the following equivalent conditions.

1. Every ideal is **finitely generated** (f.g.)

$$I = \langle f_1, \ldots, f_m \rangle = Rf_1 + \cdots + Rf_m.$$

2. **ACC** (Ascending Chain Condition) on ideals:

$$I_1 \subset I_2 \subset \cdots \text{ ideals } \Rightarrow I_N = I_{N+1} = \cdots \text{ eventually all become equal.}$$

Equations of affine varieties

Corollary Any vanishing set $V = \mathbb{V}(I)$ is the common zero locus in k^n of a **finite** number of polynomials:

$$V = \{a \in \mathbb{A}^n \colon f_1(a) = \ldots = f_m(a) = 0\}.$$

Proof Use the Hilbert Basis Theorem: take a set of generators $f_1, \ldots, f_m$ of the ideal I. So

$$I = \langle f_1, \ldots, f_m \rangle \lhd R.$$

Then clearly $f(a) = 0$ for all $f \in I$ if and only if $f_i(a) = 0$ for all $i = 1, \ldots, m$. □

We will often refer to $f_1 \ldots, f_m$ as the "equations of V", even though the set of equations is not really well defined.

On Noetherian rings

Easy proposition If R is Noetherian, any quotient of R is also Noetherian.

Corollary A finitely generated k-algebra S is Noetherian.

Easy proposition If R is Noetherian, any ideal of I is contained in a maximal ideal $\mathfrak{m}$.

Proof Keep adding elements; eventually you must get to a maximal ideal by the ACC. $\square$

This statement is true in fact in arbitrary rings, but the proof is harder and requires Zorn's Lemma.

Hilbert's Weak Nullstellensatz

Theorem (Weak Nullstellensatz) Assume that k is algebraically closed. Then every maximal ideal of the ring $R = k[x_1, \ldots, x_n]$ is of the form $\mathfrak{m}_a \lhd R$ for some $a = (a_1, \ldots, a_n) \in k^n$.

This fails over fields that are not algebraically closed.

Example Let $k = \mathbb{R}$, $R = \mathbb{R}[x]$, and $I = \langle x^2 + 1 \rangle$.
Then $I = \ker \psi$ for $\psi \colon R \to \mathbb{C}$ given by $f \mapsto f(i)$.

So R/I is a field, and in particular $I \lhd R$ is maximal. But clearly I is a principal ideal not generated by degree one polynomial(s).

Corollary $\mathbb{V}(I) = \emptyset \Leftrightarrow 1 \in I \Leftrightarrow I = R$.

Proof If $1 \notin I$ then I is a proper ideal, so it lies inside some maximal ideal $\mathfrak{m}$.
By the Weak Nullstellensatz $\mathfrak{m} = \mathfrak{m}_a$ for some $a \in \mathbb{A}^n$.
But $I \subset \mathfrak{m}_a$ implies $\mathbb{V}(I) \supset \mathbb{V}(\mathfrak{m}_a) = \{a\}$. $\square$

The Zariski topology on k^n

The **Zariski topology** on k^n is defined by declaring that the closed sets are the sets of the form $\mathbb{V}(I)$.

Easy proposition This is indeed a topology!

Proof Use easy properties of vanishing sets listed above! □

The open sets of the topology look as follows:

$$\begin{aligned} U_I &= k^n \setminus \mathbb{V}(I) \\ &= k^n \setminus (\mathbb{V}(f_1) \cap \cdots \cap \mathbb{V}(f_m)) \\ &= (k^n \setminus \mathbb{V}(f_1)) \cup \cdots \cup (k^n \setminus \mathbb{V}(f_m)) \\ &= D_{f_1} \cup \cdots \cup D_{f_m}, \end{aligned}$$

where $I = \langle f_1, \ldots, f_m \rangle$ and the D_{f_i} are called the **basic open sets**

$$D_f = k^n \setminus \mathbb{V}(f) = \{a \in k^n : f(a) \neq 0\}.$$

Let $\mathbb{A}^n$ be **affine n-space**, the topological space k^n with the Zariski topology.

The Zariski topology on $\mathbb{A}^1$

Example. $\mathbb{A}^1 = k$ has the following closed sets: $\emptyset, \mathbb{A}^1$, and all finite subsets of $\mathbb{A}^1$.

Proof Let $I \lhd R = k[x]$. As R is a PID, we have $I = \langle f \rangle$ for some $f \in R$. If f is not constant, it has a finite set of roots.

Conversely, any finite subset of k is clearly the root set of some polynomial f.

$\square$

Correspondingly, the open sets in $\mathbb{A}^1$ are $\emptyset, \mathbb{A}^1$, and the complement of any finite set of points.

Some things to observe:

- This topology is not Hausdorff, since any two non-empty open sets intersect.
- The open sets are dense, as the only closed set with infinitely many points is $\mathbb{A}^1$ (note k is infinite).

The Zariski topology on $\mathbb{A}^2$

The following statement is not obvious; see Problem Sheet 1.

Example. $\mathbb{A}^2 = k$ has the following closed sets: $\emptyset, \mathbb{A}^2$, and finite unions of

- plane curves given by equations $p(x, y) = 0$, and
- points of $\mathbb{A}^2$.