C3.4 Algebraic Geometry

Lecture 3: The coordinate ring and morphisms of affine varieties

Balázs Szendrői, University of Oxford, Michaelmas 2020

The coordinate ring of an affine variety

Let $X \subset \mathbb{A}^n$ be a (nonempty) affine variety. Let $R = k[x_1, \ldots, x_n]$ as usual.

Definition The **coordinate ring** of X is the quotient ring

$$k[X] = R/\mathbb{I}(X).$$

Interpretation The ring $k[X]$ is the ring of polynomial functions **on the variety X**. Polynomial functions on $\mathbb{A}^n$ which are zero everywhere along X do not change the value of such a function **on X**.

Properties

- $k[X]$ is a **finitely generated, so Noetherian, k-algebra**.
- As discussed before, $\mathbb{I}(X)$ is always a radical ideal, so $k[X]$ is a **reduced ring** (it has no nilpotent elements).
- $k[X]$ is an **integral domain** if and only if X is irreducible.
- $k[X]$ is a **field** if and only if $X = \{a\}$ a point (k is algebraically closed!).

The coordinate ring of an affine variety: examples

Let $X \subset \mathbb{A}^n$ be an affine variety. Let $R = k[x_1, \ldots, x_n]$. The **coordinate ring** of X is the quotient ring

$$k[X] = R/\mathbb{I}(X).$$

Examples

- For $X = \mathbb{A}^n$, we have $\mathbb{I}(X) = 0$ so $k[X] \cong R$.
- For $a \in \mathbb{A}^n$ a point, we have $\mathbb{I}(a) = \mathfrak{m}_a$, the maximal ideal at a, and as we saw before,
  $$k[a] = R/\mathfrak{m}_a \cong k.$$
- For $X = V_f$ a hypersurface defined by $f \in R$, we get
  $$k[V_f] = R/\langle f \rangle,$$
  the quotient of R by the principal ideal generared by f.

The ideal-subvariety correspondence for an affine variety

Consider a ring R and an ideal $I \lhd R$. Set $S = R/I$ and let $q\colon R \to S$ be the quotient map. Easy result in commutative algebra:

Proposition There there is an inclusion-preserving one-to-one correspondence between ideals $I \subset \tilde{J} \lhd R$ and ideals $J \lhd S$, given by $\tilde{J} = q^{-1}(J)$.

Since the correspondence is inclusion-reversing, it preserves maximal ideals. We thus recover a more general version of the ideal-variety correspondence.

Corollary There are order-reversing bijections between affine subvarieties $Y \subset X$ and ideals $J \lhd k[X]$.

$$
\begin{aligned}
(Y \subset X) &\mapsto (\mathbb{I}_X(Y) \lhd k[X]) \\
(\mathbb{V}(\tilde{J}) \subset X) &\leftarrow\!\shortmid (J \lhd k[X]) \\
\{\text{irreducible subvarieties } Y \subset X\} &\leftrightarrow \{\text{prime ideals } \mathfrak{p} \lhd k[X]\} \\
\{\text{points } a \in X\} &\leftrightarrow \{\text{maximal ideals } \mathfrak{m}_a \lhd k[X]\}
\end{aligned}
$$

Subvarieties and quotient algebras

It is worth spelling this correspondence out one more time from another point of view.

For any subvariety $Y \subset X \subset \mathbb{A}^n$, we have $\mathbb{I}(X) \subset \mathbb{I}(Y) \lhd R = k[x_1, \ldots, x_n]$. We also have coordinate rings

$$k[X] = R/\mathbb{I}(X)$$

and

$$k[Y] = R/\mathbb{I}(Y).$$

From $\mathbb{I}(X) \subset \mathbb{I}(Y)$, we get a surjection

$$k[X] \twoheadrightarrow k[Y],$$

with kernel $\mathbb{I}_X(Y) \lhd k[X]$, the ideal of **functions on X that vanish on Y**.

So closed affine subvarieties (closed inclusions of affine varieties) $Y \subset X$ correspond to surjective ring homomorphisms between their coordinate rings.

A basis for the Zariski topology on general affine varieties

We emphasise once more a basic aspect of the Zariski topology on a general affine variety $X \subset \mathbb{A}^n$.

Take $f \in k[X]$, then we get, by the correspondence above, a closed subvariety

$$\mathbb{V}(f) \subset X \subset \mathbb{A}^n$$

and hence a Zariski open set, a so-called **basic open set**

$$D_f = X \setminus \mathbb{V}(f) = \{p \in X \colon f(p) \neq 0\}.$$

Proposition The basic open sets form a basis of the Zariski topology on X: any open set is a **finite** union of basic open sets.

Proof Any open set in X is of the following form, for $I = \langle f_1, \ldots, f_m \rangle \lhd k[X]$:

$$\begin{aligned} U &= X \setminus \mathbb{V}(I) \\ &= X \setminus (\mathbb{V}(f_1) \cap \cdots \cap \mathbb{V}(f_m)) \\ &= (X \setminus \mathbb{V}(f_1)) \cup \cdots \cup (X \setminus \mathbb{V}(f_m)) \\ &= D_{f_1} \cup \cdots \cup D_{f_m}. \qquad \square \end{aligned}$$

Basic open sets: a restatement

It seems worth spelling out the following, equivalent form of this. Let $X \subset \mathbb{A}^n$ be an affine variety.

Corollary Let $p \in X$ and

$$p \in U \subset X$$

a Zariski neighbourhood of p. Then there exists $f \in k[X]$ such that

$$p \in D_f \subset U$$

is also a Zariski neighbourhood of p.

Proof As proved above, any open set is covered by (finitely many) such. □

While simple, this is often very useful: when considering neighbourhoods of points in affine varieties, we can restrict to basic affine opens.

Morphisms between affine varieties

A function $F : \mathbb{A}^n \to \mathbb{A}^m$ is a **morphism** (or **polynomial map**) if it is defined by polynomials:

$$F(a) = (f_1(a), \ldots, f_m(a)) \quad \text{for some } f_1, \ldots, f_m \in R = k[x_1, \ldots, x_n].$$

We sometimes colloquially write $y_j = f_j(x_i)$ for the polynomial map F.

Given two affine varieties $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$, a **morphism of affine varieties** $F : X \to Y$ is defined by the restriction of a morphism $\mathbb{A}^n \to \mathbb{A}^m$, given by

$$F(a) = (f_1(a), \ldots, f_m(a)) \quad \text{for some } f_1, \ldots, f_m \in k[X].$$

Note that indeed, $f_i \in k[X]$ is enough information to define F **on** X.

$F : X \to Y$ is an **isomorphism**, if F is a morphism and there is an inverse morphism $G : Y \to X$ with $F \circ G = \text{id}$, $G \circ F = \text{id}$.

Examples of morphisms between affine varieties

(E1) **Linear projections** Let $n \geq m$, and $F\colon \mathbb{A}^n \to \mathbb{A}^m$ be defined by

$$F(a_1, \ldots, a_n) = (a_1, \ldots, a_m).$$

(E2) **Inclusions of linear subspaces** Let $n \leq m$, and $F\colon \mathbb{A}^n \to \mathbb{A}^m$ be defined by

$$F(a_1, \ldots, a_n) = (a_1, \ldots, a_n, 0, \ldots, 0).$$

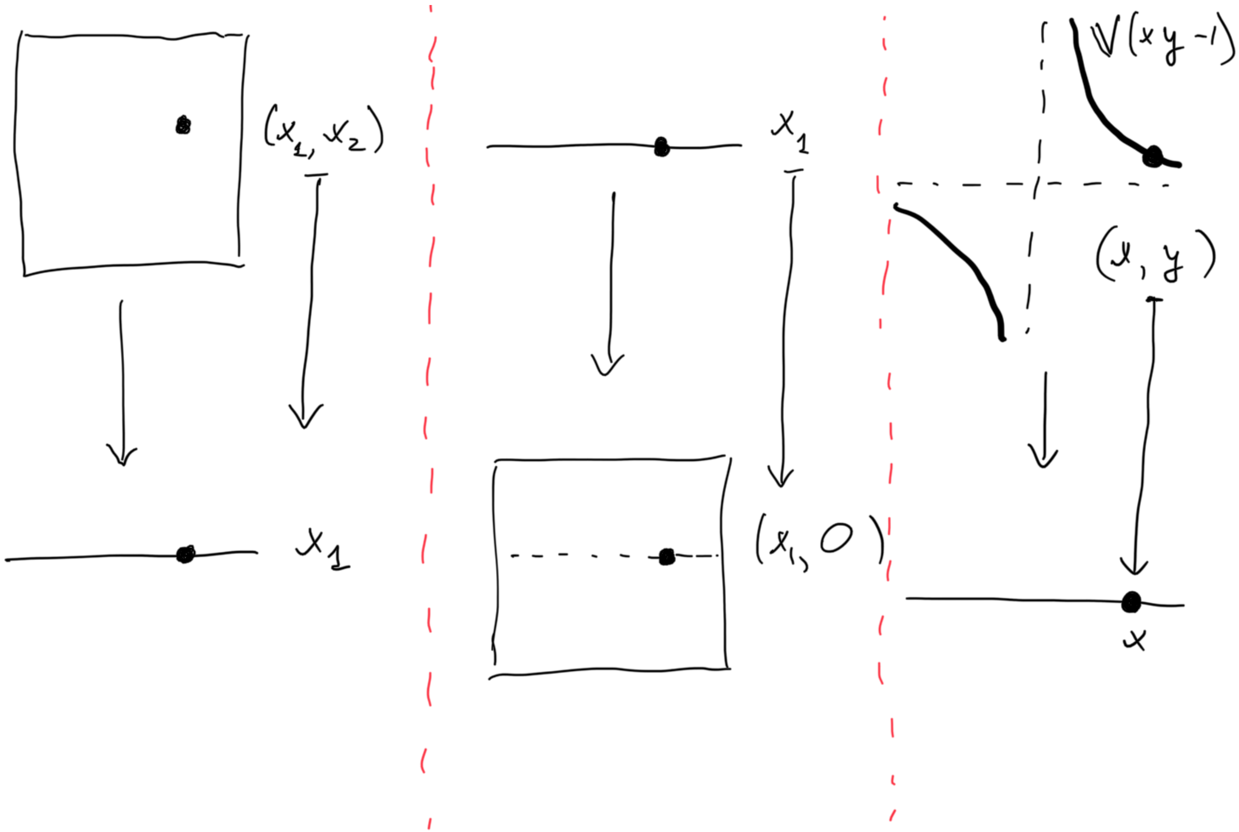
These can of course also be applied to subvarieties of $\mathbb{A}^n$. For example,

(E3) **Projection of a hyperbola** Let $X = \{xy - 1 = 0\} \subset \mathbb{A}^2$, and let $F\colon X \to \mathbb{A}^1$ be defined by

$$F(x, y) = x.$$

Note that in this example, the image set is $\mathbb{A}^1 \setminus \{0\}$ so the **image of a morphism does not need to be an affine subvariety**.

Examples of morphisms between affine varieties



More examples of morphisms between affine varieties

(E4) **A polynomial map defined on the affine line** Let $F\colon \mathbb{A}^1 \to \mathbb{A}^3$ be defined by

$$F(t) = (t, t^2, t^3).$$

(E5) **The map to the cuspidal cubic** Let $X = \mathbb{A}^1$, and

$$Y = \{x^3 - y^2\} \subset \mathbb{A}^2.$$

Let $F\colon X \to Y$ be defined by

$$f(t) = (t^2, t^3).$$

This is obviously a polynomial map $\mathbb{A}^1 \to \mathbb{A}^2$; to make it into a map to Y we just need to check that the image is contained in Y. This is easy.

Morphisms and coordinate rings

Given affine varieties $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$, write $k[X] = k[x_1, \ldots, x_n]/\mathbb{I}(X)$, $k[Y] = k[y_1, \ldots, y_m]/\mathbb{I}(Y)$. Define

$$\mathrm{Hom}(X, Y) = \{\text{morphisms } F\colon X \to Y\}$$

and

$$\mathrm{Hom}_{k\text{-alg}}(k[Y], k[X]) = \{k\textbf{-algebra homs } k[Y] \to k[X]\}.$$

Theorem (Fundamental theorem of affine algebraic geometry) There is a one-to-one correspondence

$$\begin{array}{rcl} \mathrm{Hom}(X, Y) & \longleftrightarrow & \mathrm{Hom}_{k\text{-alg}}(k[Y], k[X]) \\ F : X \to Y & \mapsto & F^* : k[Y] \to k[X] \\ \varphi^* : X \to Y & \leftarrow\!\shortmid & \varphi : k[Y] \to k[X] \end{array}$$

with a morphism F given in coordinates by $F(a) = (f_1(a), \ldots, f_m(a))$ and the maps given by

$$\begin{aligned} F^*(y_j) &= f_j(x_1, \ldots, x_n), \\ \varphi^*(a) &= (\varphi(y_1)(a), \ldots, \varphi(y_m)(a)). \end{aligned}$$

More on morphisms and coordinate rings

The statement is almost a "tautology": it is easy to check that the maps in the statement are indeed mutual inverses (see Lecture Notes).

Note that in particular, we can think of

$$k[X] = \mathrm{Hom}(X, \mathbb{A}^1),$$

since an element $f \in k[X]$ is nothing but a function $a \mapsto f(a)$ from X to $\mathbb{A}^1$.

In this language, given a morphism $F\colon X \to Y$, the map

$$F^* : k[Y] \to k[X]$$

can be thought of as composition of F with a map to $\mathbb{A}^1$

$$F^* : \mathrm{Hom}(Y, \mathbb{A}^1) \to \mathrm{Hom}(X, \mathbb{A}^1), \; g \mapsto F^* g = g \circ F$$

and in particular, it indeed "goes backwards".

How the correspondence works in practice

It is easy to run the correspondence in practice!

Work with coordinates $\{x_1, \dots, x_n\}$ on $\mathbb{A}^n$ and $\{y_1, \dots, y_m\}$ on $\mathbb{A}^m$. A morphism $F\colon \mathbb{A}^n \to \mathbb{A}^m$ is given in these coordinates by a bunch of polynomial expressions

$$y_j = f_j(x_1, \dots, x_n), \quad j = 1, \dots, m,$$

giving the map

$$F(x_1, \dots, x_n) = (f_1(x_i), f_2(x_i), \dots, f_m(x_i)).$$

You can read the same formulae as

$$y_j \mapsto f_j(x_1, \dots, x_n), \quad j = 1, \dots, m$$

which immediately defines a ring morphism

$$k[y_1, \dots, y_m] \longrightarrow k[x_1, \dots, x_n],$$

which is exactly the dual map F^*!

How the correspondence works in practice

More generally, for affine varieties $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$, the story is exactly the same: have F defined by

$$F(x_1, \ldots, x_n) = (f_1(x_i), f_2(x_i), \ldots, f_m(x_i))$$

and dually F^* defined by

$$y_j \mapsto f_j(x_1, \ldots, x_n), \quad j = 1, \ldots, m,$$

with the following conditions:

- The map F must have the property that $a \in X$ must be mapped to $F(a) \in Y$.
- Dually, the map F^* must descend to a ring map

$$k[Y] = k[y_1, \ldots, y_m]/\mathbb{I}_Y \longrightarrow k[X] = k[y_1, \ldots, x_n]/\mathbb{I}_X.$$

Morphisms and coordinate rings: Example 1

(E1) **Linear projection** Let $n \geq m$, and $F\colon \mathbb{A}^n \to \mathbb{A}^m$ be defined by

$$F(a_1, \ldots, a_n) = (a_1, \ldots, a_m).$$

This corresponds to the k-algebra homomorphism

$$\begin{array}{rcll} F^* & : & k[y_1, \ldots, y_m] & \to & k[x_1, \ldots, x_n] \\ & & y_i & \mapsto & x_i. \end{array}$$

Note that

- The image of F is dense in $Y = \mathbb{A}^n$ (in fact it is equal to $\mathbb{A}^m$ in this case).
- F^* is injective.

Morphisms and coordinate rings: Example 2

(E2) **Inclusions of linear subspaces** Let $n \leq m$, and $F\colon \mathbb{A}^n \to \mathbb{A}^m$ be defined by

$$F(a_1, \ldots, a_n) = (a_1, \ldots, a_n, 0, \ldots, 0).$$

This corresponds to the k-algebra homomorphism

$$\begin{aligned} F^* \; : \; k[y_1, \ldots, y_m] &\to k[x_1, \ldots, x_n] \\ y_i &\mapsto \begin{cases} x_i & \text{if } i \leq n \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Note that

- F is a closed inclusion.
- F^* is surjective.

Morphisms and coordinate rings: Example 3

(E3) **Projection of a hyperbola** Let $X = \{xy - 1 = 0\} \subset \mathbb{A}^2$, and let $F\colon X \to \mathbb{A}^1$ be defined by

$$F(x, y) = x.$$

This corresponds to the k-algebra homomorphism

$$\begin{array}{rcrcl} F^* & : & k[\mathbb{A}^1] = k[t] & \to & k[X] = k[x,y]/\langle xy - 1\rangle \\ & & t & \mapsto & x. \end{array}$$

Note that

- The image $\mathbb{A}^1 \setminus \{0\}$ of F is dense in $\mathbb{A}^1$.
- F^* is injective.

Morphisms and coordinate rings: Example 4

(E4) **A polynomial map defined on the affine line** Let $F\colon \mathbb{A}^1 \to \mathbb{A}^3$ be defined by

$$F(t) = (t, t^2, t^3).$$

This corresponds to the k-algebra homomorphism

$$\begin{array}{rcrcl} F^* & : & k[\mathbb{A}^3] = k[x, y, z] & \to & k[\mathbb{A}^1] = k[t] \\ & & x & \mapsto & t \\ & & y & \mapsto & t^2 \\ & & z & \mapsto & t^3 \end{array}$$

Note that, once again, this is a closed inclusion, and correspondingly the ring map is surjective.

Morphisms and coordinate rings: a variant of Example 4

(E4') **A variant** Let

$$Y = \{x^2 - y = 0, xy - z = 0\} \subset \mathbb{A}^3.$$

Let $F \colon \mathbb{A}^1 \to Y$ be defined by

$$F(t) = (t, t^2, t^3).$$

Note we are using the same formulae; to make sure this is a map to Y, we need to make sure that the image points are contained in Y, but that's easy to check. This corresponds to the k-algebra homomorphism

$$\begin{array}{rcccl} F^* & : & k[Y] = k[x,y,z]/\langle xy - z, y - x^2\rangle & \to & k[\mathbb{A}^1] = k[t] \\ & & x & \mapsto & t \\ & & y & \mapsto & t^2 \\ & & z & \mapsto & t^3 \end{array}$$

To make sure that this is well defined, we need to check that polynomials in the defining ideal of Y are mapped to zero, but that's easy.

Morphisms and coordinate rings: a variant of Example 4

(E4') **A variant, continued** On the other hand, let $G\colon Y \to \mathbb{A}^1$ be defined by $G(x,y,z) = x$, so

$$\begin{array}{rcl} G^* \ : \ k[\mathbb{A}^1] = k[t] & \to & k[Y] = k[x,y,z]/\langle xy - z, y - x^2\rangle \\ t & \mapsto & x \end{array}$$

Compute composites:

$$(F^* \circ G^*)(t) = F^*(G^*(t)) = F^*(x) = t$$

whereas

$$\begin{array}{rcccll} (G^* \circ F^*)(x) & = & G^*(t) & = & x & \\ (G^* \circ F^*)(y) & = & G^*(t^2) & = & x^2 & = y \mod \mathbb{I}(Y) \\ (G^* \circ F^*)(z) & = & G^*(t^3) & = & x^3 & = z \mod \mathbb{I}(Y). \end{array}$$

So F^*, G^* are mutual inverses, and so are F, G. In other words, $\mathbb{A}^1$ and Y are **isomorphic affine varieties**.

General properties of the correspondence

The correspondence between morphisms $F\colon X \to Y$ between affine varieties, and dual morphisms $F^*\colon k[Y] \to k[X]$ between coordinate rings, has the following basic properties.

- F is the **inclusion of a closed affine subvariety** if and only if F^* is **surjective**.
  We call such F a **closed embedding**.
- The image of F is **a Zariski dense subvariety of Y** if and only if F^* is **injective**.
  We call such F **dominant**.

For proofs, see Problem Sheet 1. For examples, see earlier!