C3.4 Algebraic Geometry

Lecture 4: Basics of projective algebraic geometry

Balázs Szendrői, University of Oxford, Michaelmas 2020

Motivation for projective algebraic geometry

Affine algebraic geometry is a natural and beautiful subject. However, it lacks an essential “completeness” or “compactness” property. This is **not** compactness in the Zariski topology, **nor** completeness in a metric sense, but rather the issue of “including points at infinity”.

Bezout’s theorem Two non-overlapping plane curves of degrees d_1 and d_2 meet in **exactly** $d_1 \cdot d_2$ points.

- We need to work over an algebraically closed field.
- We need to count multiplicities.
- We need to **work in projective space**.

Simple special case Two distinct lines in the plane meet in **exactly** one point.

- No need for an algebraically closed field; no issue with multiplicities.
- Still need to **work in projective space**!

Ways to discuss projective space

There are different ways to approach projective space and projective geometry, all having different merits.

(1) Axiomatic approach.

(2) Approach based on linear algebra.

(3) Coordinate-based approach.

Of these, (1) is beautiful but takes a long time to get to substantial results.

We are going to use a combination of (2) and, for the most part, (3), following on from the Oxford Part A Projective Geometry course.

Definition of projective space via linear algebra

Let V be a finite-dimensional vector space over a field k. We let

$$\mathbb{P}(V) = \{\text{lines through the origin (1-dim linear subspaces) in } V\},$$

the **projective space of V**.

Let $k^* = k \setminus \{0\} =$ be the set of units (invertible elements) of k.
Since every line through the origin in V is determined by a direction (every 1-dim subspace has a generating vector, well defined up to scale), we can write

$$\mathbb{P}(V) = (V \setminus \{0\}) \,/ \sim$$

where $\sim$ is the equivalence relation on nonzero vectors in V given by

$$v \sim w \text{ if and only if } w = \lambda v \text{ for some } \lambda \in k^*.$$

Sometimes we simply write

$$\mathbb{P}(V) = (V \setminus \{0\}) \,/\, k^*$$

with the rescaling action of k^* understood.

Coordinate-based definition of projective space

It is easy to turn this description into coordinates. Choosing a basis of V, we have $V \cong k^{n+1}$, and every vector can be represented in coordinates as $v = (a_0, \ldots, a_n)$.

We get the definition of **projective n-space over k**:

$$\mathbb{P}^n_k = (k^{n+1} \setminus \{0\}) \,/\, k^* \text{ for all } \lambda \in k^*,$$

where the rescaling action is

$$(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n) \text{ for all } \lambda \in k^*.$$

Write $[a_0 : a_1 : \ldots : a_n]$ for the equivalence class of $(a_0, a_1, \ldots, a_n) \in k^{n+1} \setminus \{0\}$; so in these **projective** or **homogeneous coordinates**,

$$[a_0 : \ldots : a_n] = [\lambda a_0 : \ldots : \lambda a_n] \quad \text{for all } \lambda \in k^*.$$

Note not all a_i can be zero: $[0 : \ldots : 0]$ is not a valid choice.

Change of coordinates

Always remember: our coordinates $[x_0 : \ldots : x_n]$ on $\mathbb{P}^n$ depend on a **choice of basis** in the underlying k-vector space.

A change of basis corresponds to a matrix $A \in \mathrm{GL}(n+1, k)$, which gives a new coordinate system related to the old one by a linear transformation.

Change of basis might allow the simplification of a given polynomial (trick already used in Part A Projective Geometry course).

Example For $\mathbb{P}^1$, these changes of basis correspond to the action of **Möbius transformations**:

$$[x_0 : x_1] \mapsto [ax_0 + bx_1 : cx_0 + dx_1]$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, k).$$

The projective line and the projective plane

Consider $n = 1$.

We have

$$[a_0 : a_1] = \begin{cases} [1 : 0] & \text{if } a_1 = 0 \\ [\frac{a_0}{a_1} : 1] & \text{otherwise.} \end{cases}$$

So we get

$$\mathbb{P}^1_k = \mathbb{A}^1_k \sqcup \{\infty\}.$$

For $k = \mathbb{C}$, this is the **Riemann sphere** $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$.

Now consider $n = 2$. We have

$$[a_0 : a_1 : a_2] = \begin{cases} [a_0 : a_1 : 0] & \text{if } a_2 = 0 \\ [\frac{a_0}{a_2} : \frac{a_1}{a_2} : 1] & \text{otherwise.} \end{cases}$$

So we get

$$\mathbb{P}^2_k = \mathbb{A}^2_k \sqcup \mathbb{P}^1_k.$$

This is the description of the projective plane as the affine plane together with the **ideal line at infinity**.

Homogeneous polynomials

We continue to work with the ring of polynomials $R = k[x_0, \ldots, x_n]$, thought of as polynomials in our variables $x_0, \ldots, x_n$.
We also continue to assume k algebraically closed.

Definition A polynomial $F \in R$ is a **homogeneous polynomial of degree** d, if all the monomials $x_0^{i_0} \cdots x_n^{i_n}$ appearing in F have degree

$$d = i_0 + \cdots + i_n.$$

By convention, $0 \in R$ is homogeneous of every degree.

Lemma $F \in R$ is homogeneous of degree d, if and only if for all $\lambda \in k^*$, we have

$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n).$$

Examples All monomials $\prod_i x_i^{d_i}$ are homogeneous (of some degree); $x_0^3 + x_1 x_2^2$ is homogeneous of degree 3; $x_0^3 + x_1 x_2 + x_3$ is not homogeneous.

Evaluating polynomials

How to evaluate polynomials on projective space?

- Given a polynomial $F \in R$, it does not make sense to "evaluate F at a point $p \in \mathbb{P}^n$", since the value of f is not well-defined because of the equivalence relation.
- Given a **homogeneous** polynomial $F \in R$, it **still** does not make sense to "evaluate F at a point $p \in \mathbb{P}^n$".
- Given a **homogeneous** polynomial $F \in R$, it **does** make sense to ask whether "F vanishes at a point $p \in \mathbb{P}^n$", as

$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n)$$

  with $\lambda \in k^*$.

Homogeneous ideals

Definition An ideal $I \lhd R$ is a **homogeneous ideal**, if it is generated by homogeneous polynomials.

Notes

- There is no requirement that the generators should all be of the same degree.
- Of course I will contain lots of non-homogeneous elements; the requirement is that it should be **generated** by such.

Examples

- $I_1 = \langle x_0, \ldots, x_k \rangle \lhd k[x_0, \ldots x_n]$ is a homogeneous ideal generated by linear generators.
- $I_2 = \langle F \rangle \lhd k[x_0, \ldots x_n]$ for a homogeneous polynomial F is a principal, homogeneous ideal generated by a single homogeneous generator.
- $I_3 = \langle x_0^3 + x_1^3 + x_2^3, x_0^2 - x_1^2 \rangle \lhd k[x_0, x_1, x_2]$ is a homogeneous ideal generated by a quadric and a cubic.

Projective varieties

Definition $X \subset \mathbb{P}^n$ is a **projective variety** if

$$X = \mathbb{V}(I) = \{[a] \in \mathbb{P}^n : F(a) = 0 \text{ for all homogeneous } F \in I\}$$

for some homogeneous ideal $I \lhd k[x_0 \ldots, x_n]$.

Examples

- For $I_1 = \langle x_0, \ldots, x_k \rangle \lhd k[x_0, \ldots x_n]$,

  $$\mathbb{V}(I_1) = \{x_0 = \ldots = x_k = 0\} \subset \mathbb{P}^n$$

  is a **projective linear subspace**.

- For $I_2 = \langle F \rangle \lhd k[x_0, \ldots x_n]$ with F a homogeneous polynomial,

  $$\mathbb{V}(I_2) = \{F = 0\} \subset \mathbb{P}^n$$

  is a **projective hypersurface**.

More projective varieties

Some special cases:

- If $F \in k[x_0, x_1, x_2]$ is homogeneous of degree d,

$$\mathbb{V}(\langle F \rangle) = \{F = 0\} \subset \mathbb{P}^2$$

  is a **projective plane curve of degree d**.

- The **Fermat hypersurface** is

$$\mathbb{V}(\langle x_0^d + x_1^d + \ldots + x_n^d \rangle) = \left\{x_0^d + x_1^d + \ldots + x_n^d = 0\right\} \subset \mathbb{P}^n.$$

- Consider the vector space $V = M_n(k)$ of $n \times n$ matrices over k, with its standard basis. Then $\mathbb{P}V \cong \mathbb{P}^{n^2-1}$, with homogeneous coordinate variables being the matrix entries x_{ij}. Inside $\mathbb{P}^{n^2-1}$, we have the **determinantal hypersurface**

$$\{\det(A) = 0\} \subset \mathbb{P}^{n^2-1}.$$

  Indeed det is a homogeneous polynomial of degree n in the matrix entries.

Zariski topology on projective varieties

Definition The **Zariski topology** on $\mathbb{P}^n$ has closed sets being projective varieties $\mathbb{V}(I)$.

The Zariski topology on a projective variety $X \subset \mathbb{P}^n$ is the subspace topology, so the closed subsets of X are $X \cap \mathbb{V}(J) = \mathbb{V}(I + J)$ for any homogeneous ideal J.

Equivalently, closed subsets in X are $\mathbb{V}(K)$ for homogeneous ideals $I \subset K \triangleleft R$.

Example The Zariski closed subsets of $\mathbb{P}^1_k$ (over an algebraically closed field k) are $\mathbb{P}^1_k$ itself, $\emptyset$, and finite sets of points.

Proof This is basically a homogeneous version of the proof we had for $\mathbb{A}^1$. If F is a homogeneous polynomial of degree d, then

$$x_0^{-d} F(x_0, x_1) = f(x_1/x_0)$$

for an ordinary polynomial of the ratio x_1/x_0. The latter has a finite number of zeros in $\mathbb{A}^1_k \subset \mathbb{P}^1_k$. $\square$

Projective subvarieties

A **projective subvariety** $Y \subset X$ is a Zariski closed subset of $X = \mathbb{V}(I)$. So as before, $Y = \mathbb{V}(K)$ for a homogeneous ideal $I \subset K \lhd R$.

Examples

- For $l \geq k$,

$$\{x_0 = \ldots = x_l = 0\} \subset \{x_0 = \ldots = x_k = 0\} \subset \mathbb{P}^n$$

  is a closed subvariety inside a projective linear subspace.

- In the projective space of matrices $\mathbb{P}M_n(k) \cong \mathbb{P}^{n^2-1}$,

$$\{A^2 = 0\} \subset \{\det(A) = 0\} \subset \mathbb{P}^{n^2-1}$$

  is a projective subvariety defined by quadratic equations. Indeed, for a matrix over a field k, if $A^2 = 0$, then $\det(A) = 0$.

The affine cone

For a projective variety $X \subset \mathbb{P}^n$, the **affine cone** $\hat{X} \subset \mathbb{A}^{n+1}$ is the union of the straight lines in k^{n+1} corresponding to the points of X.

Lemma If $\emptyset \neq X = \mathbb{V}(I) \subset \mathbb{P}^n$ for some homogeneous ideal $I \subset R$, then $\widehat{X}$ is the affine variety associated to the ideal $I \subset R$:

$$\widehat{X} = \mathbb{V}(I) \subset \mathbb{A}^{n+1}.$$

This follows basically from the definitions.

Note that the Lemma does not hold if $X = \emptyset$. This will happen if $I \subset R$ does not vanish on any line in $\mathbb{A}^{n+1}$. By homogeneity of I, this forces $\mathbb{V}(I) \subset \mathbb{A}^{n+1}$ to be either $\emptyset$ or $\{0\}$, which by Nullstellensatz corresponds respectively to $I = R$ or $I = \langle x_0, \ldots, x_n \rangle$.

The ideal $I = \langle x_0, \ldots, x_n \rangle$ is called the **irrelevant ideal** of the ring R.

The vanishing ideal

Definition For any set $X \subset \mathbb{P}^n$, define the **vanishing ideal** $\mathbb{I}^h(X)$ to be the homogeneous ideal generated by the homogeneous polynomials vanishing on X:

$$\mathbb{I}^h(X) = \langle F \in R : F \text{ homogeneous}, F(X) = 0 \rangle.$$

Lemma If I is homogeneous, then $\mathbb{V}(\mathbb{I}^h(\mathbb{V}(I))) = \mathbb{V}(I)$ and $I \subset \mathbb{I}^h(\mathbb{V}(I))$.

This follows, analogously to the affine case, essentially from definitions.

Lemma For $X \subset \mathbb{P}^n$ a projective variety, we have an equality of ideals

$$\mathbb{I}^h(X) = \mathbb{I}(\widehat{X})$$

between the homogeneous vanishing ideal of X and the affine vanishing ideal of the cone over X.

Proof See notes.

The projective Nullstellensatz

Let $R = k[x_0, \ldots, x_n]$, with k algebraically closed. Let $I_{irr} = \langle x_0, \ldots, x_n \rangle \lhd R$ be the irrelevant ideal.

Theorem (Projective Nullstellensatz) For any homogeneous ideal $I \lhd R$ with $\sqrt{I} \neq I_{irr}$, we have

$$I^h(\mathbb{V}(I)) = \sqrt{I}.$$

Proof We have $\mathbb{V}_{\text{affine}}(I) \neq \{0\}$ by the affine Nullstellensatz, as $\sqrt{I} \neq I_{irr}$.

So $X = \mathbb{V}(I) \subset \mathbb{P}^n$ is non-empty, with affine cone $\widehat{X} = \mathbb{V}(I) \subset \mathbb{A}^{n+1}$.

Using Lemma above and the affine Nullstellensatz, we obtain:

$$\mathbb{I}^h(X) = \mathbb{I}(\widehat{X}) = \mathbb{I}(\mathbb{V}(I)) = \sqrt{I}.$$

□

Projective varieties and homogeneous ideals

The maps $X \mapsto \mathbb{I}^h(X)$ and $\mathbb{V}(I) \mapsfrom I$ set up $1:1$ correspondences

$$\begin{aligned}
\{\text{proj. vars. } X \subset \mathbb{P}^n\} &\leftrightarrow \{\text{homogeneous radical ideals } I \neq I_{irr}\} \\
\{\text{irred. proj. vars. } X \subset \mathbb{P}^n\} &\leftrightarrow \{\text{homogeneous prime ideals } I \neq I_{irr}\} \\
\{\text{points of } \mathbb{P}^n\} &\leftrightarrow \{\text{``maximal'' homogeneous ideals } I \neq I_{irr}\} \\
\emptyset &\leftrightarrow \{\text{the homogeneous ideal } R\}.
\end{aligned}$$

Here a point $p = [a_0 : \cdots : a_n] \in \mathbb{P}^n$ corresponds to the homogeneous ideal

$$\mathfrak{m}_p = \langle a_i x_j - a_j x_i : \text{all } i, j \rangle = \{\text{homogeneous polys vanishing at } a\}.$$

These are homogeneous ideals different from I_{irr}, maximal such with respect to inclusion.