C3.4 Algebraic Geometry

Lecture 5: Open covers and morphisms of projective varieties

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Projective space covered by affine spaces

We work in projective space $\mathbb{P}^n$ over a field k. We have homogeneous coordinates $[x] = [x_0 : \ldots : x_n] \in \mathbb{P}^n$.

Define the Zariski closed sets

$$H_i = \{[x] \in \mathbb{P}^n : x_i = 0\} \subset \mathbb{P}^n$$

and the Zariski open sets

$$U_i = \mathbb{P}^n \setminus H_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\} \subset \mathbb{P}^n.$$

Easy lemma We have

(1) There are bijections $U_i \leftrightarrow \mathbb{A}^n$ for each i.

(2) We have

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i.$$

So $\mathbb{P}^n$ is covered by the Zariski open sets $U_0, \ldots, U_n$.

Projective space covered by affine spaces

Proof of Easy Lemma For (1), the bijection between U_i and $\mathbb{A}^n$ is given by

$$\begin{aligned} \varphi_i : U_i &\rightarrow \mathbb{A}^n \\ [x] = \left[\frac{x_0}{x_i} : \cdots : \frac{x_{i-1}}{x_i} : 1 : \frac{x_{i+1}}{x_i} : \cdots : \frac{x_n}{x_i}\right] &\rightarrow \left(\frac{x_0}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_n}{x_i}\right) \end{aligned}$$

Next, for every point of $\mathbb{P}^n$, at least one of the homogeneous coordinates must be nonzero. So indeed $\mathbb{P}^n = \cup_{i=0}^n U_i$. $\square$

The quantities $\frac{x_j}{x_i}$ for fixed i are called **affine coordinates** in the **affine chart** U_i.
In calculations, it is often advisable to introduce new variables for these affine coordinates:

$$y_j = \frac{x_j}{x_i}$$

are affine coodinates, identifying $U_i \subset \mathbb{P}^n$ with affine space $\mathbb{A}^n$ with coordinates $y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$.

Projective line covered by affine spaces

Example For $n = 1$, we get

$$\mathbb{P}^1 = U_0 \cup U_1.$$

Points which have both coordinates nonzero are in both open sets.
The complements of the open sets are $H_0 = \{[0 : 1]\}$ and $H_1 = \{[1 : 0]\}$.

On $U_0 = \mathbb{A}^1$, we have the affine coordinate $y = \frac{x_1}{x_0}$. On $U_1 = \mathbb{A}^1$, we have the affine coordinate $z = \frac{x_0}{x_1}$.

For $k = \mathbb{C}$, this is the Riemann sphere as a union of two complex lines (planes), identified using stereographic projection:

$$\mathbb{P}^1_{\mathbb{C}} = \mathbb{C}_0 \cup \mathbb{C}_\infty$$

with $\mathbb{C}_0$ being the chart around $0 = [0 : 1]$ and $\mathbb{C}_\infty$ the chart around $\infty = [1 : 0]$.
The special points are the north and south poles on the Riemann sphere.

Projective varieties covered by affine varieties

For any projective variety $X \subset \mathbb{P}^n$, the sets $X \cap H_i$ are Zariski closed in it, so the sets $X \cap U_i$ are Zariski open. We have

$$X = \bigcup_{i=0}^{n} (X \cap U_i).$$

Claim $X \cap U_i \subset U_i = \mathbb{A}^n$ are affine varieties.

Proof The set $X \cap U_i \subset \mathbb{A}^n$ is defined by the vanishing of all the equations of $X \subset \mathbb{P}^n$ after the substitution $x_i = 1$ (or, what is equivalent, re-expressing the equations in the affine coordinates on U_i). $\square$

So indeed "projective varieties are locally affine": they are covered by open sets which are affine varieties. This is a very fruitful point of view.

We will call $X \cap U_i \subset \mathbb{A}^n$ the **affine charts** of the projective variety $X \subset \mathbb{P}^n$.

Affine charts of a quadric

Example Let us look at the quadric

$$Q = \{x_0^2 + x_1^2 - x_2^2 = 0\} \subset \mathbb{P}^2.$$

We have three affine charts

$$\mathbb{P}^2 = U_0 \cup U_1 \cup U_2.$$

(0) On U_0, we have $x_0 \neq 0$, and we have affine coordinates $y_1 = \frac{x_1}{x_0}, y_2 = \frac{x_2}{x_0}$. Dividing the equation by x_0^2, we get the affine quadric

$$Q \cap U_0 = \{1 + y_1^2 - y_2^2 = 0\} \subset U_0 = \mathbb{A}^2.$$

(1) On U_1, we have $x_1 \neq 0$, and we choose affine coordinates $z_0 = \frac{x_0}{x_1}, z_2 = \frac{x_2}{x_1}$. Dividing the equation by x_1^2, we get the affine quadric

$$Q \cap U_1 = \{z_0^2 + 1 - z_2^2 = 0\} \subset U_1 = \mathbb{A}^2.$$

(2) Finally on U_2 with variables t_0, t_1, we get

$$Q \cap U_2 = \{t_0^2 + t_1^2 - 1 = 0\} \subset U_2 = \mathbb{A}^2.$$

Affine charts and coordinate changes

Recall: our coordinates $[x_0 : \ldots : x_n]$ on $\mathbb{P}^n$ depended on a **choice of basis** in the underlying k-vector space.

A change of basis corresponds to a matrix $A \in \mathrm{GL}(n+1, k)$, which gives a new coordinate system related to the old one by a linear transformation.

With respect to a new coordinate system, we get a new system of hyperplanes, and new system of Zariski open sets identified with $\mathbb{A}^n$.

Example, continued Use the change of coordinates

$$X_0 = x_0, \quad X_1 = x_1 + x_2, \quad X_2 = x_1 - x_2$$

to write the equation of our quadric as

$$Q = \{X_0^2 + X_1 X_2 = 0\} \subset \mathbb{P}^2.$$

Then in this coordinate system, one of the open charts becomes

$$Q \cap U_2' = \{T_0^2 + T_1 = 0\} \subset U_2' = \mathbb{A}^2.$$

Projective closure

Given an affine variety $X \subset \mathbb{A}^n$, we can view $X \subset \mathbb{P}^n$ via:

$$X \subset \mathbb{A}^n = U_0 \subset \mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}.$$

The **projective closure** $\overline{X} \subset \mathbb{P}^n$ of X is the Zariski closure (closure in the Zariski topology) of the set $X \subset \mathbb{P}^n$.

Note that, by definition,

$$\overline{X} \cap U_0 = X \subset U_0 = \mathbb{A}^n.$$

How to find equations for $\overline{X} \subset \mathbb{P}^n$? We need to **homogenise**.

Homogenizing polynomials

Given a polynomial $f \in k[x_1, \ldots, x_n]$, write $f = f_0 + f_1 + \cdots + f_d$ where f_i are the homogeneous parts of f of degree i. The **homogenisation** of f is

$$\widetilde{f} = x_0^d f_0 + x_0^{d-1} f_1 + \cdots + x_0 f_{d-1} + f_d.$$

Easy Lemma $\widetilde{f} \in k[x_0, \ldots, x_n]$ is homogeneous of degree d, with

$$\widetilde{f}|_{x_0=1} = f.$$

Examples

- The quadric $x^2 + y^2 - 1 \in k[x, y]$ becomes the homogeneous quadric
$$x^2 + y^2 - z^2 \in k[x, y, z].$$
- The cubic $y^2 - x(x-1)(x-c) \in k[x, y]$ becomes the homogeneous cubic
$$y^2 z - x(x-z)(x-cz) \in k[x, y, z].$$

The ideal of the projective closure

Let $X = \mathbb{V}(I) \subset \mathbb{A}^n$ be an affine variety defined by an ideal $I \lhd k[x_1, \dots, x_n]$. Define

$$\widetilde{I} = \langle \widetilde{f} \colon f \in I \rangle$$

be the homogeneous ideal in $k[x_0, \dots, x_n]$ generated by the homogenisations of all elements of I (not just generators!).

Proposition The projective closure $\overline{X} \subset \mathbb{P}^n$ of X of X is

$$\overline{X} = \mathbb{V}(\widetilde{I}) \subset \mathbb{P}^n.$$

Proof See Lecture Notes. $\square$

Example The Zariski closure of the affine cubic curve

$$\{y^2 = x(x-1)(x-c)\} \subset \mathbb{A}^2$$

is the projective cubic curve

$$\{y^2 z = x(x-z)(x-cz)\} \subset \mathbb{P}^2.$$

Morphisms of projective varieties

Fix projective varieties $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$. We work with $(m+1)$-tuples of homogeneous polynomials of the projective coordinates $x_0, \ldots, x_n$ on X. Let $R = k[x_0, \ldots, x_n]$.

Definition A **morphism** $F : X \to Y$ of projective varieties is a function F such that for every $p \in X$, there is an open neighbourhood $p \in U \subset X$, and **homogeneous** polynomials $f_0, \ldots, f_m \in R$ **of the same degree**, with

$$F : U \to Y \text{ is given by } F([a]) = [f_0(a) : \cdots : f_m(a)].$$

(1) The fact that the degrees of the f_i are equal ensures that the map is well-defined:

$$\begin{aligned} F([\lambda a]) &= [f_0(\lambda a) : \cdots : f_m(\lambda a)] \\ &= [\lambda^d f_0(a) : \cdots : \lambda^d f_m(a)] \\ &= [f_0(a) : \cdots : f_m(a)] \\ &= F([a]). \end{aligned}$$

Morphisms of projective varieties

A morphism $F : X \to Y$ of projective varieties is defined on open sets $\in U \subset X$ by **homogeneous** polynomials $f_0, \dots, f_m \in R$ of the same degree:

$$F : U \to Y \text{ is given by } F([a]) = [f_0(a) : \cdots : f_m(a)].$$

(2) When constructing F, we have to make sure the f_i do not vanish simultaneously at any a.

(3) Need to make sure the image point always lands in Y, i.e. the values

$$[f_0(a) : \cdots : f_m(a)]$$

have to satisfy all the defining equations of Y.

(4) Often a single set of polynomials $f_0, \cdots, f_m$ suffices, but already in relatively simple cases more than one set of polynomials may be needed.

Definition An **isomorphism** of projective varieties is a morphism $F : X \to Y$ that has a (two-sided) inverse $G : Y \to X$.

An example: a Veronese embedding of $\mathbb{P}^1$

Consider the morphism (Veronese embedding)

$$F_1 : \mathbb{P}^1 \to \mathbb{P}^2$$

given by

$$[s : t] \mapsto [s^2 : st : t^2].$$

This is a morphism:

(1) It is defined on the whole of $\mathbb{P}^1$ by degree 2 polynomials.

(2) The polynomials do not vanish simultaneously.

(3) There are no equations to check in the image.

The example revisited

Let

$$Y = \mathbb{V}(xz - y^2) \subset \mathbb{P}^2.$$

Consider the morphism (Veronese embedding)

$$F_2 : \mathbb{P}^1 \to Y \subset \mathbb{P}^2$$

given by the same formula

$$[s : t] \mapsto [s^2 : st : t^2].$$

This is a morphism from $\mathbb{P}^1$ to Y:

(1) It is defined on the whole of $\mathbb{P}^1$ by degree 2 polynomials.

(2) The polynomials do not vanish simultaneously.

(3) The image values satisfy the defining polynomial of Y:

$$(s^2)(t^2) = (st)^2.$$

The example revisited

We want to build an inverse morphism to F_2. Define

$$G : Y \to \mathbb{P}^1$$

by $[x : y : z] \mapsto [x : y]$ if $x \neq 0$, and $[x : y : z] \mapsto [y : z]$ if $z \neq 0$.

Note the Zariski open sets $\{x \neq 0\}$ and $\{z \neq 0\}$ cover Y.

We get a well-defined map, since on the overlap $x \neq 0, z \neq 0$ we have

$$[x : y] = [xz : yz] = [y^2 : yz] = [y : z].$$

It is also easy to check that $F_2 \circ G = \mathrm{id}$, $G \circ F_2 = \mathrm{id}$. So F_2, G are inverse isomorphisms of projective varieties.

Another example: projection

Consider a projetive variety $X \subset \mathbb{P}^n$. Assume $[1 : 0 : \ldots : 0] \notin X$. Define

$$\pi : X \to \mathbb{P}^{n-1}$$

by the formula

$$[x_0 : \ldots : x_n] \mapsto [x_1 : \ldots : x_n].$$

This is a morphism:

(1) It is defined on X by degree 1 polynomials.

(2) The polynomials do not vanish simultaneously as $[1 : 0 : \ldots : 0] \notin X$.

(3) There are no equations to check in the image.

Geometric interpretation: Projection from the point $p = [1 : 0 : \ldots : 0]$, see Lecture Notes.

Last example: projective equivalences

An isomorphism $X \cong Y$ of projective varieties $X, Y \subset \mathbb{P}^n$ is a **projective equivalence**, if it is the restriction of a linear isomorphism

$$\mathbb{P}^n \to \mathbb{P}^n,\ [x] \mapsto [Ax]$$

given by an invertible $(n+1) \times (n+1)$ matrix A over k.
This morphism is induced by a linear isomorphism $\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$, $x \mapsto Ax$, where $A \in GL(n+1, k)$.

Since $[Ax] = [\lambda Ax]$, we only care about A modulo scalar matrices $\lambda \cdot I$.

Thus we only need to consider

$$A \in PGL(n+1, k) = GL(n+1, k)/k^*.$$

In particular, the group $PGL(n+1, k)$ acts on projective space $\mathbb{P}^n$ by projective equivalences.