C3.4 Algebraic Geometry

Lecture 6: Graded rings, homogenous coordinate rings and Veronese embeddings

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Graded rings

Definition Let A be a commutative ring. An **$\mathbb{N}$-grading of A** means a direct sum decomposition

$$A = \bigoplus_{d \geq 0} A_d$$

as an abelian group under addition, so that the grading is compatible with multiplication:

$$A_d \cdot A_e \subset A_{d+e}.$$

The elements in A_d are called the **homogeneous elements of degree d**.

Note every $f \in A$ is uniquely a finite sum $\sum f_d$ of homogeneous elements $f_d \in A_d$.

A **homomorphism of graded rings** $f : A \to B$ is a homomorphism of rings which respects the grading so that $f|_{A_d} : A_d \to B_d$.

In all our examples, A will be a f.g. commutative unital k-algebra with $A_0 \cong k$.

Homogeneous ideals

Suppose A is a graded ring as above. For an ideal $I \lhd A$, let

$$I_d = I \cap A_d.$$

Definition. $I \lhd A$ is a **homogeneous ideal**, if

$$I = \bigoplus_{d \geq 0} I_d.$$

Easy lemma

1. $I \lhd A$ is homogeneous if and only if I generated by homogeneous elements.
2. I is homogeneous if and only for every $f \in I$, also all homogeneous parts $f_d \in I$.
3. If is I homogeneous,

   $$I \text{ a prime ideal } \Leftrightarrow \forall\ \textbf{hgs}\ f, g \in A, fg \in I \text{ implies } f \in I \text{ or } g \in I.$$

4. Sums, products, intersections, radicals of homogeneous ideals are homogeneous.

Examples

Example 1 This is the key example. The ring

$$R = k[x_0, \ldots, x_n]$$

is graded, by declaring that $\deg x_i = 1$. The graded pieces are

$$R_d = k[x_0, \ldots, x_n]_d,$$

the spaces of homogeneous polynomials of degree d.

Example 2 We could generalise this! Declare $\deg x_i = w_i$, some positive integer. We get a **different** $\mathbb{N}$-grading on the same ring $R = k[x_0, \ldots, x_n]$. We will not use this construction in this course, but it would be an interesting direction to go in.

Quotients of graded rings

The following is still easy, but it is a key statement.

Proposition Let $A = \oplus A_d$ be a graded ring, $I \lhd A$ a homogeneous ideal. Then the quotient ring $S = R/I$ is also graded, with

$$S_d = R_d/I_d.$$

Example 3 Let $f \in k[x_0, \ldots, x_n]_d$ be a homogeneous polynomial of degree d. Then the ring

$$S = R/\langle f \rangle$$

is graded, with

$$S_e = \begin{cases} k[x_0, \ldots, x_n]_e & \text{if } e < d; \\ k[x_0, \ldots, x_n]_e / f \cdot k[x_0, \ldots, x_n]_{e-d} & \text{otherwise.} \end{cases}$$

The homogeneous coordinate ring of a projective variety

We work with $R = k[x_0, \ldots, x_n]$, with the usual grading in which $x_0, \ldots, x_n$ all have degree 1.

Consider a projective variety $X \subset \mathbb{P}^n$. The **homogeneous coordinate ring** $S(X)$ is the graded ring

$$S(X) = R/\mathbb{I}^h(X),$$

where $\mathbb{I}^h(X) \lhd R$ is the homogeneous ideal of X.

Example 1 We have

$$S(\mathbb{P}^n) = R = k[x_0, \ldots, x_n].$$

Example 2 For $X = \mathbb{V}(yz - x^2) \subset \mathbb{P}^2$, we have

$$S(X) = k[x, y, z]/(yz - x^2).$$

Reconstructing a projective variety from a graded ring

There is also a converse process. Suppose that A is a reduced, graded k-algebra with $A_0 \cong k$. Suppose also

Key assumption: A is generated by finitely many elements, all in degree 1.

Let $g_0, \ldots, g_n$ be a set of generators in degree 1 of A as a k-algebra. Consider the homomorphism of graded k-algebras

$$\varphi \colon R = k[x_0, \ldots, x_n] \to A$$

given by $\varphi(x_i) = g_i$. This is surjective, since the elements g_i generate A. Let

$$I = \ker \varphi \lhd R.$$

Then I is necessarily a graded ideal, since φ respects the gradings (as g_i is of degree 1!). I is also a radical ideal, since A is reduced.

Let $X = \mathbb{V}(I) \subset \mathbb{P}^N$. Then by the Nullstellensatz $\mathbb{I}^h(X) = I$. Hence

$$S(X) \cong R/I \cong A.$$

So the projective variety $X \subset \mathbb{P}^n$ has homogeneous coordinate ring A.

Veronese subrings of a graded ring

Let $A = \oplus_{e \geq 0} A_e$ be a graded ring, d a fixed integer. Consider

$$A^{(d)} = \bigoplus_{e \geq 0} A_{d \cdot e},$$

the d-th Veronese subring of A. This is a graded ring, with **new grading** $A_e^{(d)} = A_{d \cdot e}$.

Example Let $A = k[x, y]$ in the standard grading. Then for $d = 2$,

$$A^{(2)} = k \oplus \langle x^2, xy, y^2 \rangle \oplus \langle x^4, x^3 y, x^2 y^2, x y^3, y^4 \rangle \oplus \ldots .$$

More general example Let $R = k[x_0, \ldots, x_n]$ in the standard grading. Then

$$R^{(d)} = k \oplus k[x_0, \ldots, x_n]_d \oplus k[x_0, \ldots, x_n]_{2d} \oplus \ldots .$$

Towards the Veronese embedding

We have the graded ring

$$R^{(d)} = \bigoplus_{e \geq 0} k[x_0, \dots, x_n]_{ed}.$$

Easy Lemma $R^{(d)}$ is reduced, and generated by its degree 1 piece $R_1^{(d)} = R_d$.

Proof $R^{(d)}$ is a subring of the reduced ring R, so it is reduced itself. Also every monomial of degree $e \cdot d$ in $x_0, \dots, x_n$ is a product of d monomials of degree e (usually in many ways). $\square$

Natural question What projective variety do we get if we consider this graded ring $R^{(d)}$ and perform the construction of a projective variety from it?

The first example of a Veronese embedding

Example Consider the simplest case $n = 1$, $d = 2$. We have, as before,

$$R^{(2)} = k \oplus \langle x_0^2, x_0x_1, x_1^2 \rangle \oplus \langle x_0^4, x_0^3x_1, x_0^2x_1^2, x_0x_1^3, x_1^4 \rangle \oplus \dots .$$

The degree one piece $R_1^{(2)}$ is generated by three polynomials x_0^2, x_0x_1, x_1^2. So we consider the surjection

$$\varphi \colon k[y_0, y_1, y_2] \to R^{(2)}$$

given by $y_0 \mapsto x_0^2, y_1 \mapsto x_0x_1, y_2 \mapsto x_1^2$. The kernel is the ideal

$$\ker \varphi = \langle y_0y_2 - y_1^2 \rangle.$$

We get

$$Y = \{y_0y_2 - y_1^2 = 0\} \subset \mathbb{P}^2,$$

the image of the Veronese embedding $F_2 \colon \mathbb{P}^1 \to Y \subset \mathbb{P}^2$ we considered in Lecture 5!

Counting and listing monomials of a given degree

Fix d, n. We need to count the number of monomials of $x_0, \ldots, x_n$ of degree d.

Proposition We have

$$\dim_k k[x_0, \ldots, x_n]_d = \binom{n+d}{d}.$$

Proof Several proofs are possible. A combinatorial proof is explained in the Lecture Notes. A proof by double induction on (n, d) is left as an exercise. $\square$

In concrete calculations, it is also useful to be able to list these monomials as a linear list. The most common way to do so is the lexicographic ordering. Instead of a detailed explanation, an example should suffice.

Example For $n = 2$ and $d = 3$, we have $\dim_k k[x_0, x_1, x_2]_2 = \binom{5}{3} = 10$.
A lexicographically ordered basis of $k[x_0, x_1, x_2]_2$ is

$$\{x_0^3, x_0^2 x_1, x_0^2 x_2, x_0 x_1^2, x_0 x_1 x_2, x_0 x_2^2, x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3\}.$$

Definition of the Veronese embedding

Fix d, n as before. Let $N = \binom{n+d}{d} - 1$.

Define **the d-th Veronese map on $\mathbb{P}^n$** to be the projective morphism

$$\begin{array}{rccc} \nu_d \ : & \mathbb{P}^n & \longrightarrow & \mathbb{P}^N \\ & [x_0 : \ldots : x_n] & \mapsto & [\ldots : x^I : \ldots] \end{array}$$

where the index set runs over all monomials $x^I = x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n}$ of degree $d = i_0 + \cdots + i_n$, in other words a basis of the space $k[x_0, \ldots, x_n]_d$.

The image of ν_d inside $\mathbb{P}^N$ is called a **Veronese variety**.

Note: indeed ν_d is a projective morphism, as it is defined by homogeneous polynomials of the same degree d, not all zero as the monomials include $x_0^d, \ldots, x_n^d$.

Equations satisfied by the image of the Veronese map

We now want to write down equations that the image of ν_d satisfies.
The morphism is defined by considering all degree d monomials x^I of $x_0, \ldots, x_n$.
Consider multi-indices of type $(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ with $i_0 + \cdots + i_n = d$. Note that if for multi-indices I, J, K, L, we have $I + J = K + L$, then

$$x^I x^J = x^K x^L.$$

This means that, considering variables z_I on $\mathbb{P}^N$ for all possible multi-indices I, we get

$$\mathrm{image}(\nu_d) \subset \mathbb{V}(z_I z_J - z_K z_L) \subset \mathbb{P}^N$$

for all multi-indices I, J, K, L with $I + J = K + L$.

Example For the familiar case $n = 1$, $d = 2$, we have

$$(x_0 x_1)^2 = (x_0^2)(x_1^2)$$

which corresponds to the multi-index identity

$$(1, 1) + (1, 1) = (2, 0) + (0, 2).$$

The main result

Theorem The Veronese map is a closed embedding $\nu_d \colon \mathbb{P}^n \hookrightarrow \mathbb{P}^N$. It defines an isomorphism between $\mathbb{P}^n$ and its image

$$\begin{aligned} \mathrm{Image}(\nu_d) &= \mathbb{V}(\langle z_I z_J - z_K z_L : I + J = K + L \rangle) \\ &= \bigcap_{I+J=K+L} \mathbb{V}(z_I z_J - z_K z_L) \ \subset \ \mathbb{P}^N \end{aligned}$$

where we run over all multi-indices I, J, K, L of type $(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ with $i_0 + \cdots + i_n = d$.

Proof, Step 1. Let $Y = \bigcap_{I+J=K+L} \mathbb{V}(z_I z_J - z_K z_L) \ \subset \ \mathbb{P}^N$. The discussion on the previous page showed that $\mathrm{Image}(\nu_d) \subset Y$. So indeed we can think of ν_d as a morphism $\nu_d \colon \mathbb{P}^n \to Y$.

To finish the argument, we will write down an explicit inverse morphism for ν_d on Y.

The main result: construction of the inverse

Proof, Step 2. Fix $J = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ with $d - 1 = i_0 + \cdots + i_n$, and denote $J_\ell = (j_0, \ldots, j_\ell + 1, \ldots, j_n)$; these indices now sum to d. Let

$$\varphi_J : Y \setminus Z_J \to \mathbb{P}^n \text{ defined by } [\ldots : z_I : \ldots] \mapsto [z_{J_0} : z_{J_1} : \ldots : z_{J_n}]$$

which is a well-defined morphism away from the closed set Z_J where all $z_{J_\ell} = 0$. These morphisms φ_J fit together to a morphism

$$\varphi\colon Y \to \mathbb{P}^N.$$

Indeed for two such J, J', notice $J_\ell + J'_{\ell'} = J_{\ell'} + J'_\ell$ (this equals $J + J'$ plus add 1 in the two slots ℓ, ℓ').
Hence $z_{J_\ell} z_{J'_{\ell'}} = z_{J_{\ell'}} z_{J'_\ell}$, and thus $[z_{J_\ell} : z_{J_{\ell'}}] = [z_{J'_\ell} : z_{J'_{\ell'}}]$ and so finally

$$\varphi_J([z]) = \varphi_{J'}([z]).$$

The main result: checking the inverse properties

Proof, Step 3. We show that ν_d, φ are inverse morphisms, finally proving

$$\mathbb{P}^n \cong Y \subset \mathbb{P}^N.$$

Notice

$$\varphi_J \circ \nu_d([x]) = [x^{J_0} : \ldots : x^{J_n}] = [x_0 : \ldots : x_n],$$

as we can just rescale by $1/x^J$.

Conversely, one can also check

$$\nu_d \circ \varphi_J([z_I]) = [z_I].$$

For the somewhat intricate combinatorial argument to prove this, see Lecture Notes. $\square$

Rational normal curves

The d-th Veronese embedding of $\mathbb{P}^1$ For $n = 1, d$ arbitrary, we get a closed inclusion

$$\nu_d \colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$$

defined by $[x_0 : x_1] \mapsto [x_0^d : x_0^{d-1}x_1 : \ldots : x_1^d]$.

The image is traditionally called the **rational normal curve of degree d**.

In this case, the equations can be written in the following attractive form.

$$\nu_d(\mathbb{P}^1) = \left\{ \operatorname{rank} \begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix} \leq 1 \right\} \subset \mathbb{P}_d.$$

Indeed, the condition that the rank of the given matrix is at most 1 is captured by the vanishing of 2×2 determinants

$$\det \begin{pmatrix} y_i & y_j \\ y_{i+1} & y_{j+1} \end{pmatrix} = y_i y_{j+1} - y_j y_{i+1}.$$

These are precisely the quadrics from the Theorem above.

The Veronese surface

The second Veronese embedding of $\mathbb{P}^2$ For $n = 2, d = 2$, we get a closed inclusion

$$\nu_2 \colon \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$$

defined by $[x_0 : x_1 : x_2] \mapsto [x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2]$.

The image is traditionally called the **Veronese surface**.