C3.4 Algebraic Geometry

Lecture 7. Products and the Segre embedding

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Products in algebraic geometry

Given two algebraic varieties X and Y (say affine or projective), we would like to define what it means to form the product $X \times Y$ of X and Y.
Reasonable requests from any kind of product:

(1) $X \times Y$ should be the same kind of object as X and Y are (affine or projective variety).

(2) There should be projection maps $X \times Y \to X$ and $X \times Y \to Y$ which are morphisms of the appropriate structures.

(3) The set of points of $X \times Y$ should be in one-to-one correspondence with pairs (x, y) with $x \in X$ and $y \in Y$.

A brief reflection shows that for $X = \mathbb{A}^n$ and $Y = \mathbb{A}^m$, the natural choice is $X \times Y = \mathbb{A}^{n+m}$, with affine projection maps. However, this also shows

(4) We cannot require the Zariski topology on $X \times Y$ to be the product topology; that would have too few open sets.

Ways of forming products in algebraic geometry

There are different points of view on how to form products in algebraic geometry.

(1) Algebraic: tensor product

(2) Categorical: using the **universal property** of the product

(3) Geometric: via defining equations and the Segre embedding

(1) and (2) are briefly discussed in the Lecture Notes (non-examinable). We will follow approach (3).

The product of two affine varieties

Here we take as our starting point the idea that $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ which is certainly very natural.

Start with general affine varieties given by equations as follows:

$$X = \mathbb{V}(f_1, \ldots, f_N) \subset \mathbb{A}^n, \quad f_j = f_j(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$$

and

$$Y = \mathbb{V}(g_1, \ldots, g_M) \subset \mathbb{A}^m, \quad g_i = g_i(y_1, \ldots, y_m) \in k[y_1, \ldots, y_m].$$

Definition Define the product $X \times Y$ to be

$$X \times Y = \mathbb{V}(f_1, \ldots, f_N, g_1, \ldots, g_M) \subset \mathbb{A}^{n+m}$$

using the coordinate ring $k[\mathbb{A}^{n+m}] = k[x_1, \ldots, x_n, y_1, \ldots, y_m]$.

Checking properties of the product

For $X = \mathbb{V}(f_1, \ldots, f_N) \subset \mathbb{A}^n$ and $Y = \mathbb{V}(g_1, \ldots, g_M) \subset \mathbb{A}^m$, we define

$$X \times Y = \mathbb{V}(f_1, \ldots, f_N, g_1, \ldots, g_M) \subset \mathbb{A}^{n+m}.$$

Properties:

(1) $X \times Y$ thus defined is indeed an affine variety: it is the zero-set of a finite set of polynomials in $\mathbb{A}^{n+m}$.

(2) There is a projection morphism

$$p_1 \colon X \times Y \to \mathbb{A}^n$$

given by $p_1(x_i, y_j) = (x_i)$. If $(x, y) \in X \times Y$, then the x coordinates satisfy the polynomials f_j and so we can consider p_1 as a morphism of affine varieties

$$p_1 \colon X \times Y \to X.$$

The argument for the other projection is the same.

Checking properties of the product

For $X = \mathbb{V}(f_1, \ldots, f_N) \subset \mathbb{A}^n$ and $Y = \mathbb{V}(g_1, \ldots, g_M) \subset \mathbb{A}^m$, we define

$$X \times Y = \mathbb{V}(f_1, \ldots, f_N, g_1, \ldots, g_M) \subset \mathbb{A}^{n+m}.$$

(3) The argument used to show (2) also shows that indeed the set of points of $X \times Y$ is in one-to-one correspondence with pairs (x, y) with $x \in X$ and $y \in Y$.

Example Let

$$X = \mathbb{V}(x^2 - 1) \subset \mathbb{A}^1$$

and

$$Y = \mathbb{V}(y^2 - 4) \subset \mathbb{A}^1.$$

Then both X and Y consist of two points $x = \pm 1$, respectively $y = \pm 2$. Their product

$$X \times Y = \mathbb{V}(x^2 - 1, y^2 - 4) \subset \mathbb{A}^2$$

consists of four points $(\pm 1, \pm 2)$.

The algebra of products

A different point of view: assume $X = \mathbb{V}(I) \subset \mathbb{A}^n$ and $Y = \mathbb{V}(J) \subset \mathbb{A}^m$ for $I \lhd k[x_1, \ldots, x_n]$ and $J \lhd k[y_1, \ldots, y_m]$.
Consider

$$K = I \cdot k[x_i, y_j] + J \cdot k[x_i, y_j] \lhd k[x_1, \ldots, x_n, y_1 \ldots, , y_m].$$

Then it is immediate from the definitions that

$$X \times Y = \mathbb{V}(I + J).$$

For those who know tensor products, this leads to the following statement.

Proposition The coordinate rings of X, Y and their product are related by

$$k[X \times Y] = k[X] \otimes_k k[Y].$$

Proof : See notes.

How to form the product of projective varities?

What should we do when $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$?

In the affine case, we used $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$.

However, in the projective case $\mathbb{P}^n \times \mathbb{P}^m$ does not have an obvious projective variety structure. It certainly is **not** $\mathbb{P}^{n+m}$.

We need to start by realising the set $\mathbb{P}^n \times \mathbb{P}^m$ as a projective variety.

The Segre map

The **Segre map** is the function

$$\begin{aligned}\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m &\to \mathbb{P}^{(n+1)(m+1)-1} = \mathbb{P}^{nm+n+m} \\ ([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) &\mapsto [x_0 y_0 : x_0 y_1 : \cdots : x_i y_j : \cdots : x_n y_m]\end{aligned}$$

The **Segre variety** is

$$\Sigma_{n,m} = \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^{nm+n+m}$$

At the moment, it is best to regard $\sigma_{n,m}$ as a map of sets, and $\Sigma_{n,m}$ as a subset of $\mathbb{P}^{nm+n+m}$.

The Segre map in matrix language

One way to understand the Segre map: in the language of matrices.

Its target $\mathbb{P}^{nm+n+m}$ is the projective space of $(n+1)\times(m+1)$ matrices.

It maps a pair of vectors $[x_i]$, $[y_j]$ to the matrix whose (i,j) entry is $z_{ij} = x_i y_j$. So

- all the columns are multiples of $[x]$, with proportionality constants y_j;
- all the rows are multiples of $[y]$, with proportionality constants x_i.

Its image is exactly the locus of rank-1 matrices (up to scale).

The Serge variety in $\mathbb{P}^3$

Example Look at the first interesting case. We have

$$\sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

mapping $([x:y],[u:v]) \mapsto [xu:xv:yu:yv]$ or perhaps

$$([x:y],[u:v]) \mapsto \left[\begin{pmatrix} xu & xv \\ yu & yv \end{pmatrix}\right]$$

From an image point $[z_{00} : z_{01} : z_{10} : z_{11}]$, we can recover

$$[x:y] = [ux:uy] = [z_{00} : z_{10}]$$

if $u \neq 0$, or

$$[x:y] = [vx:vy] = [z_{01} : z_{11}]$$

if $v \neq 0$. So in all cases, we can recover $[x:y]$.
Similarly, we can also recover $[u:v]$.

The image of $\sigma_{1,1}$ is defined by the equation $z_{00}z_{11} - z_{10}z_{01} = 0$ inside $\mathbb{P}^3$.

Properties of the Serge map

Theorem The Segre map is an injection

$$\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{(n+1)(m+1)-1} = \mathbb{P}^{nm+n+m}.$$

Its image is exactly the subvariety

$$\mathbb{V}(z_{ij}z_{k\ell} - z_{kj}z_{i\ell} : 0 \leq i < k \leq n, 0 \leq j < \ell \leq m) \subset \mathbb{P}^{nm+n+m}.$$

Proof Everything follows from the matrix interpretation. From an image point we can recover both $[x]$ and $[y]$ as the one-dimensional column space, respectively row space.

The stated equations simply say that a matrix is of rank 1 if and only if all its 2×2 minors

$$\begin{vmatrix} z_{ij} & z_{i\ell} \\ z_{kj} & z_{k\ell} \end{vmatrix}$$

vanish. $\square$

We can now **define** the projective variety $\mathbb{P}^n \times \mathbb{P}^m$ to be the projective subvariety of $\mathbb{P}^{nm+n+m}$ given by the above equations. The Theorem above shows that its points are indeed in bijection with points of the set $\mathbb{P}^n \times \mathbb{P}^m$.

Given $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ projective varieties, we can now proceed as before: consider

$$\sigma_{n,m}(X \times Y) \subset \mathbb{P}^{nm+n+m}.$$

Proposition The above set is a projective subvariety of $\mathbb{P}^{nm+n+m}$, whose points are in bijection with $X \times Y$.

Proof Say $X = \mathbb{V}(F_1, \ldots, F_N)$, and $Y = \mathbb{V}(G_1, \ldots, G_M)$.
The set $\sigma_{n,m}(X \times Y)$ can then be written as

$$\Sigma_{n,m} \cap \mathbb{V}(F_k(z_{0j}, \ldots, z_{nj}), G_\ell(z_{i0}, \ldots, z_{im}) : \text{ all } k, \ell, i, j) \subset \mathbb{P}^{nm+n+m}.$$

As $\sigma_{n,m}$ is an injection, the rest follows. $\square$

Projective varieties from matrices

Using the language of matrices, we can begin to write down some interesting chains of projective varieties.

Example Let $\mathbb{P}\mathrm{M}_k(3) \cong \mathbb{P}^8$ be the projective space of 3×3 matrices over k. Then there is a chain of subvarities

$$\Sigma_{2,2} \subset \Delta \subset \mathbb{P}^8.$$

Here

$$\Delta = \{[A] \colon \det A = 0\} \subset \mathbb{P}^8$$

is a projective cubic hypersurface, defined by the determinant polynomial. Also $\Sigma_{2,2} \cong \mathbb{P}^2 \times \mathbb{P}^2$ is the Segre variety in $\mathbb{P}^8$.

Note that $\Delta \subset \mathbb{P}^8$ is the (projective) locus of singular matrices: 3×3 matrices of rank at most 2. This locus contains the locus of matrices of rank 1, which is exactly $\Sigma_{2,2}$.