C3.4 Algebraic Geometry

Lecture 8. Grassmannians and Flag Varieties

Balázs Szendrői, University of Oxford, Michaelmas 2020

The definition of the Grassmannian

We first define the Grassmannian as a set.

Definition The **Grassmannian** of d-planes in k^n is

$$\mathrm{Gr}(d, n) = \{d\text{-dimensional vector subspaces } U \subset k^n\}.$$

Examples

- $\mathrm{Gr}(1, n) = \mathbb{P}(k^n) \cong \mathbb{P}^{n-1}$.
- $\mathrm{Gr}(n-1, n) = \mathbb{P}((k^n)^*) \cong \mathbb{P}^{n-1}$.
- Projective duality more generally says

$$\mathrm{Gr}(d, n) \cong \mathrm{Gr}(n-d, n).$$

As the initial examples suggest, $\mathrm{Gr}(d, n)$ may always have the structure of a projective variety. We will show this, and work out some equations, at least in a simple case.

Describing the Grassmannian by linear algebra

The vector space k^n comes with a fixed basis $\{e_1, \dots, e_n\}$.

Suppose that $U \subset k^n$ is a d-dimensional linear subspace. Choose a basis $u_1, \dots, u_d$ for it. We can arrange these as rows[1] of a $d \times n$ matrix

$$A_U \in M_k(d, n)$$

which has maximal rank d (as $u_1, \dots, u_d$ form a basis).

If we change the basis in U using a change-of-basis matrix $P \in \mathrm{GL}(d)$, we get a new matrix

$$B_U = PA_U \in M_k(d, n),$$

also of rank d, that corresponds to the same subspace U.

Proposition There is a one-to-one correspondence

$$\mathrm{Gr}(d, n) \leftrightarrow \mathrm{Max}(d, n)/\mathrm{GL}(d)$$

where $\mathrm{Max}(d, n) \subset M_k(d, n)$ is the subset of matrices of maximal rank d.

[1] This is a correction compared to the lecture, where I had mistakenly spoken of columns.

Describing the Grassmannian by linear algebra

Proposition There is a one-to-one correspondence

$$\mathrm{Gr}(d,n) \leftrightarrow \mathrm{Max}(d,n)/\mathrm{GL}(d)$$

where $\mathrm{Max}(d,n) \subset M_k(d,n)$ is the subset of matrices of maximal rank d.

Example When $d=1$, this says

$$\mathrm{Gr}(1,n) \leftrightarrow \mathrm{Max}(1,n)/\mathrm{GL}(1).$$

Here $\mathrm{Max}(1,n)$ is row vectors except the zero vector, so

$$\mathrm{Max}(1,n) = k^n \setminus 0.$$

Also

$$\mathrm{GL}(1) = k^*.$$

We recover

$$\mathbb{P}^{n-1} \leftrightarrow (k^n \setminus 0)/k^*.$$

Coordinates on the Grassmannian

Suppose we want to define a map

$$\mathrm{Gr}(d,n) \to \mathbb{P}^N$$

to some projective space. What should our coordinate functions be?
We need to define functions on $\mathrm{Max}(d,n)$ which are independent of the particular vector representative chosen, i.e. which are **invariant under the action of the change of basis group $GL(d)$**.

Answer: consider **$d \times d$ minors (determinants of $d \times d$ submatrices) of the matrix A_U attached to the subspace U**.
We need to choose d columns among the n columns in total, so there are $\binom{n}{d}$ such minors.
The collection of minors is invariant under the group action: if we left-multiply A_U by a $d \times d$ matrix P, all minors change by $\det P$, so the projective point they describe does not change.
Also not all minors are zero by the maximal rank assumption.

The Plücker embedding

Theorem Associating to a subspace U the collection of $d \times d$ minors of its representing matrix A_U gives a closed embedding

$$\mathrm{Gr}(d, n) \hookrightarrow \mathbb{P}^{\binom{n}{d}-1},$$

whose image is a projective subvariety of $\mathbb{P}^{\binom{n}{d}-1}$. In particular, $\mathrm{Gr}(d, n)$ is projective.

We will not discuss the general proof. Instead, we will look at the simplest nontrivial example.

If $n \leq 3$, all Grassmannians are either points or projective spaces. So the first interesting case is when $n = 4$ and $d = 2$.

Plücker relation(s)

Example Let $d = 2, n = 4$. The Plücker map is

$$\mathrm{Gr}(2,4) \to \mathbb{P}^5$$

given in matrix form by

$$\left[\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}\right] \mapsto [af - be : ag - ce : ah - de : bg - cf : bh - df : ch - dg].$$

Proposition The Plücker map is an embedding

$$\mathrm{Gr}(2,4) \hookrightarrow \mathbb{P}^5$$

with image

$$\mathbb{V}(y_0 y_5 - y_1 y_4 + y_2 y_3) \subset \mathbb{P}^5.$$

Proof It is easy to check that the image of the Plücker map satisfies this quadratic relation, the **Plücker relation**. To complete the proof, we consider affine charts.

An affine open set in the Plücker embedding

The Plücker map of $\mathrm{Gr}(2,4)$ has image contained in

$$\mathbb{V}(y_0y_5 - y_1y_4 + y_2y_3) \subset \mathbb{P}^5.$$

Let us consider the affine open set

$$\mathrm{Gr}(2,4)_0 \subset \mathbb{V}(y_0y_5 - y_1y_4 + y_2y_3) \cap \{y_0 \neq 0\} \subset \{y_0 \neq 0\} = \mathbb{A}^5 \subset \mathbb{P}^5.$$

In the matrix coordinates used before, this means $af - be \neq 0$. So for the corresponding 2-dimensional subspace $U \subset k^4$, the first two columns of the matrix A_U are linearly independent.
This means that we can pre-multiply the matrix A_U by a unique change-of-basis matrix P so that the first two columns become the standard basis vectors of a 2-dimensional vector space.
We get an equivalence of matrices

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & C & D \\ 0 & 1 & G & H \end{pmatrix}.$$

An affine open set in the Plücker embedding

For $U \in \mathrm{Gr}(2,4)_0$, we have the representing matrix

$$\begin{pmatrix} 1 & 0 & C & D \\ 0 & 1 & G & H \end{pmatrix}.$$

We can then read off the affine Plücker coordinates of this subspace U as

$$U \mapsto (G, H, -C, -D, CH - DG).$$

We deduce the following:

- The affine Plücker relation

  $$y_5 - y_1 y_4 + y_2 y_3 = 0$$

  indeed holds.

- There are no further equations involving the Plücker coordinates.

- In this open set, we can recover the subspace U uniquely from its Plücker image.

An affine open set in the Plücker embedding

Considering all such affine charts, we deduce that over the whole Grassmannian $\mathrm{Gr}(2,4)$, the Plücker map is an embedding, and its image equals

$$\mathbb{V}(y_0y_5 - y_1y_4 + y_2y_3) \subset \mathbb{P}^5.$$

□

From the preceding argument, we also deduce

Corollary The affine open set $\mathrm{Gr}(2,4)_0$ is isomorphic to affine four-space $\mathbb{A}^4$.

Proof The projection $\mathbb{A}^5 \to \mathbb{A}^4$ to the first four coordinates, and the inclusion $\mathbb{A}^4 \to \mathbb{A}^5$ defined by

$$(x_1, x_2, x_3, x_4) \to (x_1, x_2, -x_3, -x_4, x_2x_3 - x_1x_4),$$

give inverse isomorphisms between $\mathrm{Gr}(2,4)_0 \subset \mathbb{A}^5$ and $\mathbb{A}^4$. □

Irreducibility

Theorem The Grassmannian $\mathrm{Gr}(d, n)$ is an irreducible variety.

We need a Lemma.

Lemma Let $\mathrm{GL}_n(k) \subset \mathbb{A}^{n^2}$ be the space of invertible linear matrices inside the affine space of all $n \times n$ matrices over k. Then $\mathrm{GL}_n(k)$ is an irreducible affine variety.

Proof Let $\Delta \in k[\mathbb{A}^{n^2}]$ be the determinant polynomial on the space of matrices. Then

$$\mathrm{GL}_n(k) = D_\Delta,$$

the principal open subset defined by the non-vanishing of Δ. The first statement is then a general instance of the phenomenon that a basic open set in an affine variety is affine. See Lecture 12 later!
Also $\mathrm{GL}_n(k) = D_\Delta \subset \mathbb{A}^{n^2}$ is dense, as affine space $\mathbb{A}^{n^2}$ is irreducible. A dense subset of an irreducible variety must itself be irreducible. This concludes the proof. □

Irreducibility: the proof

Proof of Theorem We define a surjective polynomial map

$$\varphi\colon \mathrm{GL}_n(k) \to \mathrm{Gr}(d, n).$$

A surjective image of an irreducible variety must be irreducible, so the existence of the map φ proves the irreducibility of $\mathrm{Gr}(d, n)$.

Inside the vector space k^n with fixed basis $\{e_1, \ldots, e_n\}$, let $W = \langle e_1, \ldots, e_d \rangle$ be a reference d-dimensional subspace.

Suppose that $V \subset k^n$ is an arbitrary d-dimensional linear subspace. Choose a basis $v_1, \ldots, v_d$ for it, and complete to a basis $v_1, \ldots, v_n$ of k^n.

Then $A \in \mathrm{GL}_n(k)$ with columns v_i will map W to V. This defines the surjective polynomial map φ, thus concluding the proof. □

Irreducibility: a comment

You may have noticed that this proof was a little bit cheating: the morphism

$$\varphi\colon \mathrm{GL}_n(k) \to \mathrm{Gr}(d,n)$$

is not a morphism of affine varieties, nor a morphism of projective varieties. We will fill in the gap in Lecture 11, where we will discuss morphisms of quasi-projective varieties.

Flag Varieties

A generalization: fix integers $0 < d_1 < \ldots < d_s < n$. A **flag of type** $(d_1, \ldots, d_s)$ is a nested sequence of subspaces

$$V_1 \subset \cdots \subset V_s \subset k^n, \dim V_i = d_i.$$

The **Flag variety** $\mathrm{Flag}(d_1, \ldots, d_s, n)$ is

$$\mathrm{Flag}(d_1, \ldots, d_s, n) = \{\text{flags } V_1 \subset \cdots \subset V_s \subset k^n, \dim V_i = d_i\}.$$

We have maps

$$\mathrm{Flag}(d_1, \ldots, d_s, n) \to \mathrm{Gr}(d_i, n)$$

defined by $(V_1 \subset \cdots \subset V_s \subset k^n) \mapsto (V_i \subset k^n)$.
Using a combination of Plücker maps, we then get

$$\mathrm{Flag}(d_1, \ldots, d_s, n) \hookrightarrow \mathbb{P}^{\binom{n}{d_1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{d_s}-1},$$

which can be used to show that $\mathrm{Flag}(d_1, \ldots, d_s, n)$ also has the structure of a projective variety. A similar argument to the one above shows that in fact $\mathrm{Flag}(d_1, \ldots, d_s, n)$ is irreducible as well.

Flag Varieties: an example

Example Suppose that $\{v_1, \dots, v_n\}$ is an (ordered) basis of k^n. Then we get a (full) flag

$$\langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \dots \subset \langle v_1, \dots, v_{n-1} \rangle \subset k^n.$$

This gives a point

$$[v_1, \dots, v_n] \in \mathrm{Flag}(1, \dots, n-1, n)$$

of the **full flag variety** of k^n. This allows us to handle things like choice of basis in a more invariant, and geometric, way.

Flag Varieties: another example

The first non-trivial case is $\mathrm{Flag}(1,2,3)$, parametrising full flags in a 3-dimensional vector space. This can be characterised as follows:

$$\mathrm{Flag}(1,2,3) = \{(l,H) \colon l \subset H\} \subset \mathbb{P}(k^3) \times \mathbb{P}(((k^3)^*) \cong \mathbb{P}^2 \times \mathbb{P}^2.$$

Here $l \subset k^3$ and $H \subset k^3$ are lines, respectively hyperplanes, through the origin.

Combining with the Serge embedding, we see that there is a chain of embeddings

$$\mathrm{Flag}(1,2,3) \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8.$$