C3.4 Algebraic Geometry

Lecture 9. Dimension theory

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Chains in varieties and dimension

Let X be a variety (affine or projective). A **chain of length** m means a strict chain of inclusions

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_m \subset X,$$

where each $X_i \subset X$ is an **irreducible** subvariety.
For example, one can start with $X_0 = \{p\}$ a point of X.
If X is irreducible, then one can end with $X_m = X$.

Definition

- The **local dimension** $\dim_p X$ of X at a point $p \in X$ is the maximum over all lengths of chains starting with $X_0 = \{p\}$.
- The **dimension** of X is the maximum of the lengths of all chains,

$$\dim X \;=\; \max_m (\exists \text{ chain } X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_m) \;=\; \max_{p \in X} \dim_p X.$$

Say X has **pure dimension** if the $\dim_p X$ are equal for all $p \in X$.

Some elementary examples

Example 1 The chain $\mathbb{A}^0 = \{0\} = \mathbb{V}(x_1, \ldots, x_n) \subset \mathbb{A}^1 = \mathbb{V}(x_2, \ldots, x_n) \subset \cdots \subset \mathbb{A}^{n-1} = \mathbb{V}(x_n) \subset \mathbb{A}^n$ shows $\dim \mathbb{A}^n \geq n$.

Example 2 Similarly, $\dim \mathbb{P}^n \geq n$.

Fact (to be discussed later) $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$, both varieties being of pure dimension.

Example 3 Consider $X = \mathbb{V}(xy, xz) \subset \mathbb{A}^3$.
As we saw earlier, X is a union of the (y, z)-plane and the x-axis.
We have $\dim_p X = 2$ at points p in the plane, and $\dim_p X = 1$ at other points.

Example 4 An (affine or projective) variety X of dimension $\dim X = 0$ is a finite set of points.

Proof Let $p \in X$ and X_0 an irreducible component of X containing p.
Suppose $X_0 \neq \{p\}$. Then we have the chain $\{p\} \subsetneq X_0$ in X, so we would have to have $\dim X \geq \dim_p X \geq 1$.
So each of the finitely many irreducible components of X must be a point.

Chains in commutative algebra

Let A be a ring (commutative with unit). A **chain of length** m means a strict chain of inclusions

$$\wp_0 \supsetneqq \wp_1 \supsetneqq \cdots \supsetneqq \wp_{m-1} \supsetneqq \wp_m$$

where each $\wp_i \lhd A$ is a prime ideal.
One can start with a maximal ideal $\wp_0 = \mathfrak{m} \subset A$. If A is an integral domain, one can end with $\wp_m = \{0\}$.

Fact For A Noetherian, the descending chain condition holds for prime ideals, i.e. there are no chains of infinite length.

Definition The **height** $\mathrm{ht}(\wp)$ of a prime ideal $\wp \lhd A$ is the maximal length of a chain with $\wp_0 = \wp$:

$$\mathrm{ht}(\wp) = \max_m \left(\exists \text{ chain } \wp \supsetneqq \wp_1 \supsetneqq \cdots \supsetneqq \wp_{m-1} \supsetneqq \wp_m\right).$$

Dimension in commutative algebra

Definition The **Krull dimension** of A is the maximum height of all maximal ideals $\mathfrak{m} \lhd A$ (equivalently, prime ideals):

$$\dim A = \max \operatorname{ht}(\mathfrak{m} \colon \mathfrak{m} \lhd A \text{ maximal}).$$

Examples

1. A field $A = k$ has $\dim A = 0$.
2. If A is a PID but not a field, then $\dim A = 1$.
3. The chain $(x_1, \ldots, x_n) \supset (x_1, \ldots, x_{n-1}) \supset \cdots \supset (x_1) \supset \{0\}$ shows $\dim k[x_1, \ldots, x_n] \geq n$.

Some key results from commutative algebra

The following results are proved in commutative algebra courses.

Theorem (Krull's principal ideal theorem, Hauptidealsatz) For any Noetherian ring A, if $f \in A$ is neither a zero-divisor nor a unit, then

$$\mathrm{ht}((f)) = 1.$$

Theorem (Krull's height theorem) For any Noetherian ring A, and $\langle f_1, \ldots, f_m \rangle \neq A$,

$$\mathrm{ht}(\langle f_1, \ldots, f_m \rangle) \leq m.$$

So the height $\mathrm{ht}(\wp)$ is at most the number of generators of $\wp$.

Dimension and transcendence degree

Definition Consider a field extension K/k. Then the transcendence degree $\mathrm{trdeg}_k K$ of K/k is the maximum number of elements of K which are algebraically independent over k (i.e. they satisfy no polynomial relations with coefficients in k).

A key result in field theory says that $\mathrm{trdeg}_k K = m$ if and only if there are m algebraically independent elements $\alpha_1, \ldots, \alpha_m \in K$ with $K/k(\alpha_1, \ldots, \alpha_m)$ a finite extension.

Theorem Let A be a finitely generated k-algebra which is an integral domain. Let $\mathrm{Frac}(A)$ be the field of fractions of A. Then

$$\dim A = \mathrm{trdeg}_k \mathrm{Frac}(A).$$

Additivity for prime ideals

Our last result from commutative algebra is the following.

Theorem (additivity for prime ideals) Let A be a finitely generated k-algebra which is an integral domain. Then for every prime ideal $\wp \lhd A$, we have

$$\mathrm{ht}(\wp) + \dim A/\wp = \dim A.$$

Compare this with the following, much easier result.

Proposition For any Noetherian ring A, let $f \in A$, neither a zero-divisor nor a unit. Then

$$\dim A/(f) \leq \dim A - 1,$$

but **equality frequently fails**.

Proof Lift a chain of prime ideals from $A/(f)$ to A, and use that $\mathrm{ht}((f)) = 1$ by the Hauptidealsatz. $\square$

A key deduction

Corollary We have $\dim k[x_1, \ldots, x_n] = n$.

Proof 1 We know the maximal ideals are $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$, so they have height at most n by Krull's height theorem.
So $\dim k[x_1, \ldots, x_n] \leq n$.
On the other hand, we noted the easy direction $\dim k[x_1, \ldots, x_n] \geq n$ above.
$\square$

Proof 2 We have

$$\dim k[x_1, \ldots, x_n] = \operatorname{trdeg}_k k(x_1, \ldots, x_n).$$

But this latter quantity is clearly n, using the key result in field theory mentioned before. $\square$

Algebraic and geometric dimension coincide

Theorem If $X \subset \mathbb{A}^n$ is an affine variety, then the (geometric) dimension of X and the (Krull) dimension of its coordinate ring agree:

$$\dim X = \dim k[X].$$

Proof By Hilbert's Nullstellensatz, there is an inclusion-reversing bijection between irreducible subvarieties $X_j \subset X$ and prime ideals $\wp_j \lhd k[X]$, given by $\wp_j = \mathbb{I}(X_j)$ and $X_j = \mathbb{V}(\wp_j)$. This gives a bijection between maximal chains. $\square$

Similarly,

Theorem For a projective variety $X \subset \mathbb{P}^n$, $\dim X$ equals the maximal length of chains of homogeneous prime ideals of its projective coordinate ring $S(X)$ which do not contain the irrelevant ideal $(x_0, \ldots, x_n)$.
In particular, $\dim X = \dim \hat{X} - 1$, where $\hat{X} \subset \mathbb{A}^{n+1}$ is the affine cone of X.

Basic properties of dimension

Proposition If X, Y are isomorphic affine, respectively projective varieties, then they have the same dimension.

Proof This is clear from the geometric definition: isomorphisms map maximal chains of irreducible subvarieties to each other.

Caveat Note if $X \cong Y$ are **affine** varieties, then $k[X] \cong k[Y]$ and so we can also see immediately that Krull dimensions coincide.
However, this is **not** true in the projective case; an isomorphism of projective varieties does not induce an isomorphism between homogeneous coordinate rings!

Proposition If X, Y are affine, respectively projective varieties, then

$$\dim(X \times Y) = \dim X + \dim Y.$$

Proof is sketched in the Lecture Notes.

Basic properties of dimension

Proposition If $X \subset \mathbb{A}^n$ is an irreducible affine variety, and $\overline{X} \subset \mathbb{P}^n$ its projective closure, then

$$\dim X = \dim \overline{X}.$$

Proof One direction $\dim X \leq \dim \overline{X}$ is clear: given a chain of irreducible subvarieties in X, we can take their projective closure to get a chain of irreducible subvarieties of $\overline{X}$. The proof of the full result is omitted. $\square$

Corollary If $X \subset \mathbb{P}^n$ is an irreducible projective variety, and $U \subset \mathbb{A}^n$ an affine open subset of X, then

$$\dim X = \dim U.$$

Linear subspaces

A **linear subspace** of $\mathbb{P}^n = \mathbb{P}(k^{n+1})$ is a projectivisation $L = \mathbb{P}(U)$ of a vector subspace $U \subset k^{n+1}$.

If $\dim_k U = m + 1$ in the sense of linear algebra, then

$$U = \langle v_0, \dots, v_m \rangle \subset V.$$

Changing basis in k^{n+1} so that these vectors belong to the basis, we can write

$$L = \mathbb{P}U = \{x_{m+1} = x_{m+2} = \dots = x_n = 0\} \subset \mathbb{P}^n.$$

So the homogeneous ideal defining L is

$$\mathbb{I}_L^h = \langle x_{m+1}, x_{m+2}, \dots, x_n \rangle \lhd k[x_0, \dots, x_n].$$

So its homogeneous coordinate ring is

$$S(L) \cong k[x_0, \dots, x_m]$$

and so

$$\dim L = \dim k[L] - 1 = m$$

the projective (linear) dimension of L.

Hypersurfaces

Theorem For an irreducible affine variety $X \subset \mathbb{A}^n$, we have $\dim X = n - 1$ if and only if

$$X = \mathbb{V}(f) \subset \mathbb{A}^n$$

for an irreducible $f \in R = k[x_1, \ldots, x_n]$.
The analogous result also holds for $X \subset \mathbb{P}^n$ an irreducible projective variety and f homogeneous in $k[x_0, \ldots, x_n]$.

Proof $(\Rightarrow)$: $\dim X = n - 1 \Rightarrow \mathbb{I}(X) \neq (0) \Rightarrow \exists f \neq 0 \in \mathbb{I}(X)$.
Since $\mathbb{I}(X)$ is prime, it must contain an irreducible factor of the factorization of f. So we can assume that f is irreducible, hence prime, in the UFD R.
Then $X \subset \mathbb{V}(f) \subsetneq \mathbb{A}^n$ is a chain of irreducibles, $\dim X = n - 1$ and $\dim \mathbb{A}^n = n$, thus we must have $X = \mathbb{V}(f)$.

$(\Leftarrow)$: As f is irreducible, $\wp = \langle f \rangle$ is a prime ideal. By the Hauptidealsatz, $\mathrm{ht}(\wp) = 1$. Now use the Theorem on additivity for prime ideals. $\square$

Such (affine or projective) varieties are called **hypersurfaces**.

Examples

Example 1 Let $f \in k[x_0, x_1, x_2]$ be a non-constant homogeneous polynomial of degree d. Let

$$C = \mathbb{V}(f) \subset \mathbb{P}^2.$$

Then we have $\dim C = 1$ by the Hauptidealsatz.
Indeed, $C \subset \mathbb{P}^2$ is a **plane curve**, a hypersurface in $\mathbb{P}^2$.

Example 2 Consider the Segre embedding $\sigma_{1,1} \colon \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. Its image is given by

$$\sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{V}(x_0x_3 - x_1x_2) \subset \mathbb{P}^3.$$

This is a hypersurface of dimension $2 = 1 + 1 = \dim \mathbb{P}^1 + \dim \mathbb{P}^1$.

An example from the Veronese embedding

Example 3 Consider the Veronese embedding $\nu_3 \colon \mathbb{P}^1 \to \mathbb{P}^3$.
The image is given by

$$X = \nu_3(\mathbb{P}^1) \cong \mathbb{V}(x_0x_2 - x_1^2, x_1x_3 - x_2^2, x_0x_3 - x_1x_2) \subset \mathbb{P}^3.$$

Consider also

$$X \subset Y = \mathbb{V}(x_0x_3 - x_1x_2) \subset \mathbb{P}^3.$$

Then we have

- $\dim \mathbb{P}^1 = 1$.
- $\dim \nu_3(\mathbb{P}^1) = 1$, as ν_3 is an isomorphism onto its image.
- $\dim \mathbb{P}^3 = 3$.
- $\dim Y = 2$, by the Hauptidealsatz. Y is an irreducible hypersurface.
- $\dim Y - \dim X = 1$, even though $X \subset Y$ is given by **two further equations**.

Projective varieties from matrices revisited

Recall that inside the projective space $\mathbb{P}\mathrm{M}_k(3) \cong \mathbb{P}^8$ of 3×3 matrices over k, we found in Lecture 7 a chain of subvarities

$$\Sigma_{2,2} \subset \Delta \subset \mathbb{P}^8,$$

where $\Delta = \{[A]\colon \det A = 0\} \subset \mathbb{P}^8$ is a projective cubic hypersurface, defined by the determinant polynomial, and $\Sigma_{2,2} \cong \mathbb{P}^2 \times \mathbb{P}^2$ is the Segre variety in $\mathbb{P}^8$.

In this chain, we have $\dim \mathbb{P}^8 = 8$, so $\dim \Delta = 7$ as it is a hypersurface. Also $\dim \Sigma_{2,2} = \dim \mathbb{P}^2 \times \mathbb{P}^2 = 2 + 2 = 4$. So, marking dimensions, the chain becomes

$$\Sigma_{2,2}^4 \subset \Delta^7 \subset \mathbb{P}^8.$$

Grassmannians

Recall

$$\mathrm{Gr}(2,4) \cong \mathbb{V}(y_0y_5 - y_1y_4 + y_2y_3) \hookrightarrow \mathbb{P}^5.$$

This being a hypersurface, we get $\dim \mathrm{Gr}(2,4) = 4$.

Note that this is compatible with the fact that we found an affine open set $U_0 \cong \mathbb{A}^4 \subset \mathrm{Gr}(2,4)$.

Proposition In general, we have

$$\dim \mathrm{Gr}(d,n) = d(n-d).$$

Sketch proof The argument given in Lecture 8 generalises to show that for arbitrary (d,n), there is an affine open set

$$U_0 \cong \mathrm{Mat}_k(d, n-d) \cong \mathbb{A}^{d(n-d)} \subset \mathrm{Gr}(d,n).$$

As $\mathrm{Gr}(d,n)$ is irreducible, its dimension agrees with any of its affine open sets, so we deduce the statement. $\square$